MA1250: INTRODUCTION TO GEOMETRY (YEAR 1) LECTURE NOTES

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1. Introduction

The word "geometry" comes to us from ancient Greek

$$\gamma\varepsilon\omega\mu\varepsilon\tau\rho\acute{\iota}\alpha = \gamma\varepsilon\omega(\text{“geo”, earth}) \ + \ \mu\varepsilon\tau\rho\acute{\iota}\alpha(\text{“metria”, measuring})$$

and as it suggests the science of geometry originates from the kind of questions that preoccupied the humanity since times immemorial – which one of two given patches of land is bigger? And in the beginning it was all about taking a contemporary equivalent of a tape measure and finding out. All measuring tools are subtly different, however, while all scientists have the same inexorable tendency to abstractify the problems they contemplate. By the Hellenistic period geometry had well established itself as a science about the *principles* of earth-measuring (traditionally contemplated upon whilst sitting in the shade of an olive tree). It studies the properties (shape, size, etc.) of points, lines and other idealised versions of real world objects and the properties of their positions relative to each other (distance, angle, etc.).

In this course, designed to serve as a gentle introduction to this venerable old subject, we aim:

- To give an overview of plane Euclidean geometry, with focus on proofs
- To introduce some basic notions of spherical geometry, emphasising its differences with Euclidean geometry.
- To practice drawing diagrams and use them as means to construct proofs
- To develop intuition and visualisation in 3 dimensions

A rigorous mathematical proof requires one to logically deduce the result you want to prove from the results you've already proven (lemmas and theorems) and the base set of assumptions you've started with (axioms). Circa 300 B.C. Euclid of Alexandria, a famous greek geometer, proposed in his immortal treatise Στοιχεῖα("Stoicheia", Elements) to rigorously deduce the whole existing body of results in plane geometry from the following set of five axioms:

(1) Through any two points there passes a unique line
(2) It is possible to extend any line segment continuously in a straight line to a larger line segment

(3) It is possible to draw a unique circle of any given radius around any given point
(4) All right angles are equal to each other
(5) If a straight line crossing two straight lines makes interior angles with them on the same side which are acute (less than a right angle) then these two lines if continued indefinitely will eventually meet on that side where angles are acute

He made a pretty good effort of it - "Elements" dictated the way geometry was taught in Europe for centuries to come. But he didn't get it completely right. In the course of deducing 6 books worth of theorems in plane geometry he made a number of implicit assumptions which didn't actually follow from these five axioms. Famously, in the very first proposition of "Elements" a pair of circles "obviously" have to intersect each other. Later on, there is a number of arguments based on superimposing one geometrical figure upon another, a procedure which has no rigorous foundation in the above axioms.

In 19th century it was realised that Euclid's axioms are actually insufficient to prove all the theorems in "Elements". Finally, in 1899 a German mathematician David Hilbert proposed a set of 20 axioms (21 originally, but one of them was later shown to follow from the other 20) from which all known theorems of Euclidean geometry can be rigorously deduced.

Due to the time constrains, we can't afford to follow either Hilbert's rigorous or even Euclid's semi-rigorous approach to plane geometry. I shall therefore cut a number of corners when proving the theorems in the course, appealing at times to intuition, and at times to the fact that due to some small advancements in mathematics since the time of 300 B.C. we now have a number of handy tools, such as the machinery of real numbers, at our disposal. You, however, should be able to prove things in exercises and exam questions fully rigorously by appealing to the axioms and the results which were proven during the lectures.

Acknowledgements: I am very grateful to David Mond, who gave this course some years before and whose excellent set of lecture notes helped me to plan out this course and served as a basis for these present notes.

2. Motivation. A comparison with the coordinate geometry

Here's a typical theorem of Euclidean geometry:

Theorem. *Given any three points which are not* **collinear**[1] *there is a unique circle which passes through all three of them.*

[1]A set of two or more points is said to be collinear if there exists a straight line which contains all of them. Note that by Euclid's first axiom such line is necessarily unique.

One can try and approach this theorem by the methods of coordinate geometry. The unique circle of radius r centered at the point (p, q) is given by the equation

$$(x-p)^2 + (y-q)^2 = r^2.$$

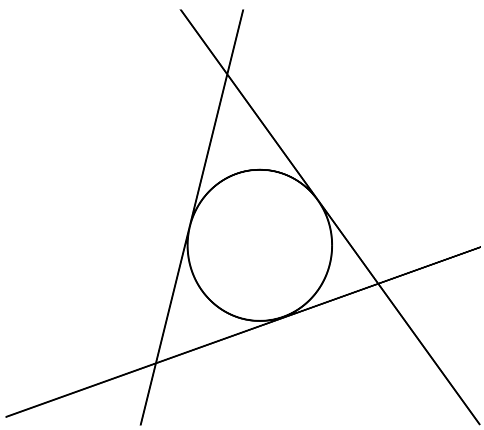
To ask that this circle passes through three given points (x_1, y_1), (x_2, y_2) and (x_3, y_3) is to ask that

$$\begin{aligned}(x_1-p)^2 + (y_1-q)^2 &= r^2\\(x_2-p)^2 + (y_2-q)^2 &= r^2\\(x_3-p)^2 + (y_3-q)^2 &= r^2\end{aligned}$$

so the problem reduces to solving the above system of equations in three unknowns: p, q and r. A system of three linear equations in three unknowns certainly has under certain conditions[2], but this is a system of quadratic equations!

This illustrates the main weakness of the coordinate geometry approach: it allows to turn any geometric problem into a bunch of equations to solve, but sometimes solving them may be harder than to solve the original problem geometrically. Indeed, suppose we ask:

Problem. *Given three lines, can we construct a circle tangent to all three of them?*



This is exactly the sort of question that Euclidean geometry helps to solve very visually and elegantly.

3. Distances and angles

Question 1: What is "the distance between two given points"?

One approach is simply to ask our plane to come equipped with an abstract function

$$d\colon \text{Plane} \times \text{Plane} \to \mathbb{R}_{\geq 0}$$

[2]its determinant has to be non-zero

which satisfies a number of axioms, such as the triangle inequality. This function then serves as a definition of distance:

point p_1 ⟶ / point p_2 ⟶ [function d] ⟶ distance between p_1 and p_2

For example, our usual notion of distance on the coordinate plane $\mathbb{R}^2$ yields the function

$$d((x_1, y_2), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

This approach leads eventually to an abstract notion of metric spaces, which can be pretty counter-intuitive to deal with.

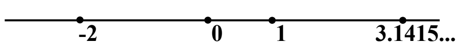
We want to model our plane geometry, to some extent, on the real world. We therefore would like our notion of distance to correspond to an intuitive idea, that the distance between two points is the length of the straight line joining them. By Euclid's first axiom there is a unique straight line passing through any two points, and therefore a unique line segment joining them. We say that the distance between two points is the length of this line segment.

Question $1\frac{1}{2}$**:** How do we measure it?

Given a line segment it is easy enough to construct a segment of twice the length. Or thrice, or four times the length. With a little ingenuity we can also construct a segment of half the length. Et cetera. This sort of thinking led men to develop the theories of integer, of rational and then of real numbers. In a sense, real numbers were invented precisely for us to be able to measure any distance along a line. Having this machinery at our disposal makes our job much easier.

Let us fix, therefore, once and for all a line segment of "unit" length. Then, given any line with a choice of the point 0 and a choice of the "positive" direction along the line, the machinery above gives us a one-to-one correspondence between the points on the line and the real numbers

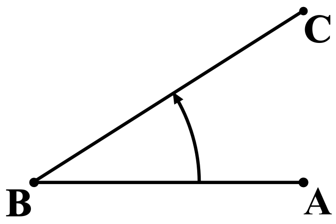
$\left\{ \begin{matrix} \text{Real numbers} \\ \mathbb{R} \end{matrix} \right\} \longleftrightarrow \left\{ \text{Points on a line} \right\}$



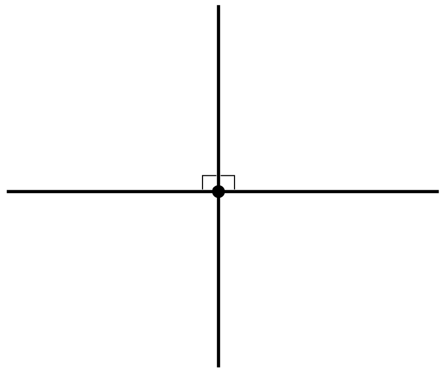
This correspondence is precisely such that the length of the line segment between the points 0 and 1 is our unit length, and the direction from 0 to 1 is our chosen positive direction.

Therefore, given any pair of points P_1 and P_2 in our plane, we can define the distance between them as follows. First, we draw the unique straight line through them. Then we choose P_1 to be the point 0 and the direction from P_1 to P_2 to be the positive direction. The real number corresponding to P_2 is then the distance between P_1 and P_2. We shall denote this by $d(P_1, P_2)$ or by $|P_1P_2|$, the latter reflecting that it is the length of the line segment P_1P_2.

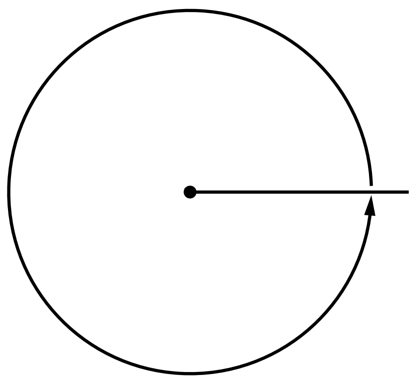
Question 2: Given three points A, B, C in the plane, what is the angle $\angle ABC$, i.e. the angle between line segment BA and line segment BC?



One might be somewhat puzzled by Euclid's fourth axiom, which asserts that all right angles are equal. To understand this, we have to consider Euclid's original definition of a right angle. Euclid said that if a straight line intersects another straight line in a way which makes two angles on one side equal then these equal angles are said to be *right angles.*



So the fourth axiom effectively asserts that all *straight angles,* angles formed by two halves of a straight line, are equal (being a sum of two right angles). Equivalently, it asserts that all angles formed by going around a point in a complete circle



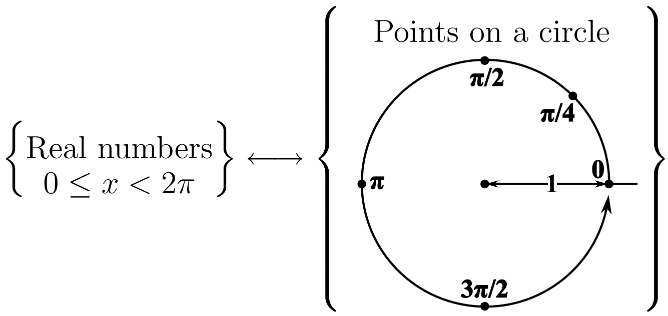
are equal (being a sum of four right angles). Therefore any two angles which are the same fraction of a complete circle are equal.

This answers the question above in a fashion which matches our intuition - the angle between line segments BA and BC is uniquely determined by the fraction of a circle around B which they cut out.

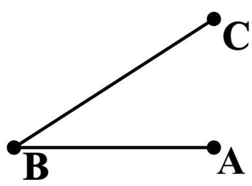
Question $2\frac{1}{2}$: How do we measure it?

We could simply define the value of the angle to be this fraction. However, above we've made a choice of a unit length and it gives us for free a choice of a "unit" circle around every point in our plane – the circle of radius 1. It is more convenient to define the value of the angle between BA and BC to be the distance we travel from BA to BC along the circumference of the unit circle around B. Since the full circumference of a unit circle is of length 2π, this effectively defines the value of an angle to be 2π times the fraction of a complete circle cut out by it.

More precisely, if we have a unit circle with a choice of the point 0 and with a choice of a direction, clockwise or counter-clockwise[3], the machinery of real numbers gives us a one-to-one correspondence between the points on the circle and the real numbers in the interval between 0 and 2π:

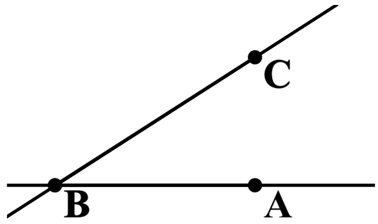


Therefore, given any three points A, B and C such that neither A nor C coincides with B,

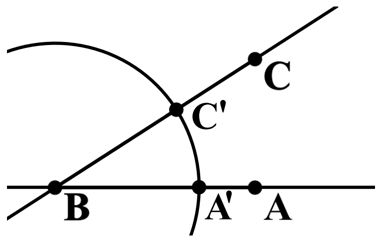


we first extend both BA and BC to straight lines

[3] By assuming, intuitively, that we have universal notions of "clockwise" and "counter-clockwise" in our plane we make implicit use of notions of orientation and of orientability, a rather sophisticated machinery of its own

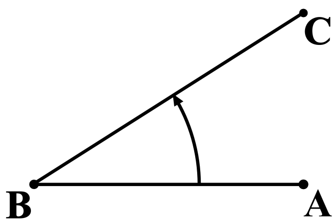


and then we draw a unit circle around B and mark by A' the point where this circle intersects the line BA on the same side of B as A. Similarly, mark by C' the point where the circle intersects the line BC on the same side of B as C.



We now define the value of $\angle ABC$ to be the real number corresponding to C' if we choose A' to be the point 0 on the circle and the direction around the circle to be anti-clockwise.

An important point: $\angle ABC$ is not the same thing as $\angle CBA$! The definition above ensures that by $\angle ABC$ we always mean the angle from BA to BC *anti-clockwise*.

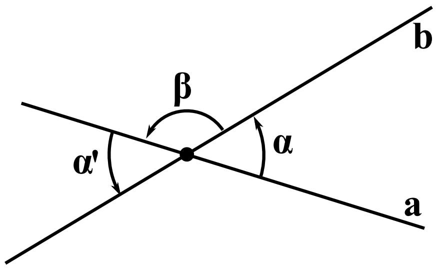


Similarly, $\angle CBA$ is the angle from BC to BA *anti-clockwise*. And since to go from BA to BC anti-clockwise and then from BC to BA anti-clockwise is to come a full circle, it follows that $\angle ABC + \angle CBA = 2\pi$, or in other words

$$\angle CBA = 2\pi - \angle ABC.$$

Finally, note that as a complete circle is an angle of 2π, any right angle is an angle of $\frac{\pi}{2}$ and every straight angle is an angle of π.

Lemma 1. *Let a and b be a pair of intersecting lines. Then the two angles where we go counter-clockwise from a to b are equal.*



Proof. We have

$$\angle\alpha + \angle\beta = \pi \quad \text{(from a straight angle formed by line } a\text{)}$$
$$\angle\beta + \angle\alpha' = \pi \quad \text{(from a straight angle formed by line } b\text{)}$$

and so we conclude that $\alpha = \alpha'$. Q.E.D. [4] □

Definition 1. Given two intersecting lines a and b we denote by $\angle ab$ the value of either of the 2 equal angles where we go from a to b anti-clockwise (cf. Lemma 1) and refer to it as *the* angle from a to b.

Definition 2. Two lines a and b are said to be perpendicular to each other if they intersect at right angles, that is $\angle ab = \angle ba = \pi/2$. We denote this by $a \perp b$.

4. Isometries and congruences

What does it mean to say that two geometrical objects are "equal"? Our intuition tells us that two objects in a plane are equal if we can move one on top of the other so that they match perfectly. Below we make this notion precise by giving a mathematical procedure for "moving one object on top of the other".

Definition 3. A `map` from the plane to the plane is a rule which sends each point in the plane to some other point in the plane. Given such a map f and a point P in the plane we denote by $f(P)$ the point where f sends P:

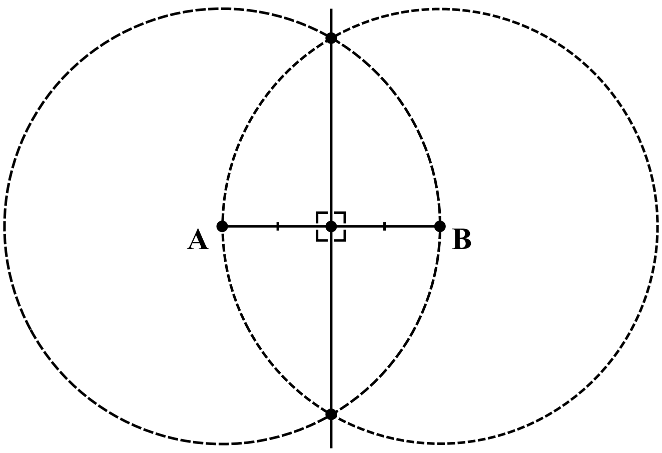
$$f\colon \text{Plane} \to \text{Plane}$$
$$P \mapsto f(P)$$

Before giving examples of this we need to establish the following notion:

Definition 4. Given any line segment AB its `perpendicular bisect` is the unique line which passes through the midpoint of AB and is perpendicular to AB.

[4] The acronym Q.E.D. which you may often see concluding a mathematical proof stands for *"quod erat demonstrandum"* which is Latin for "what was to be demonstrated". Its origin lies in the Greek phrase ὅπερ ἔδει δεῖξαι, often abbreviated ΟΕΔ, which means "what was required to be proved" and which concludes every proof in Euclid's "Elements".

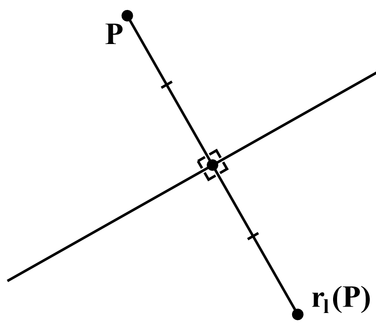
We have established in the previous section that there is a unique line passing through a given point and making a specified angle with a given line through this point. This ensures that a perpendicular bisect exists and is unique. For those unconvinced by such abstractions, here is a nice and explicit construction. Given a line segment AB draw two circles centered at A and B, both of radius $|AB|$. Then draw a straight line through the two points where these circles intersect.



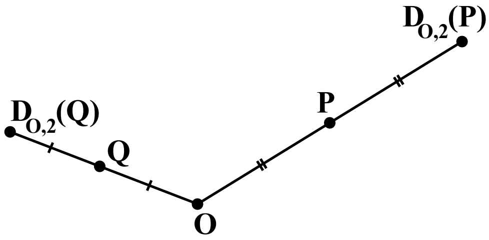
We will prove later on in the course that, as is evident on the diagram above, the resulting line is the perpendicular bisect of AB.

We now proceed to give examples of maps from the plane to itself:

Example 1. (1) Define `the reflection` r_l `in` l to be the map which sends any point P to the unique point $r_l(P)$ such that l is the *perpendicular bisect* of the line segment $Pr_l(P)$. In other words, we drop a perpendicular from P to l and then extend it by the same distance again on the other side:

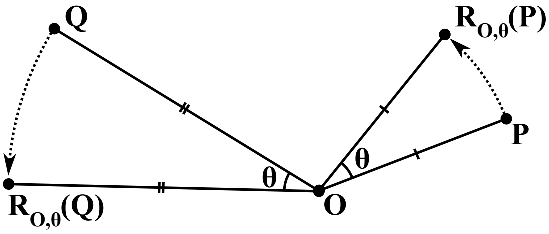


(2) Given a point O in the plane define `the dilation` $D_{O,2}$ `with centre` O `and scale factor` 2 to be the map which leaves the point O fixed and sends any point $P \neq O$ to the unique point $D_{0,2}(P)$ which lies on the continuation of the line segment OP in the direction of P at the distance $2|OP|$ from O.

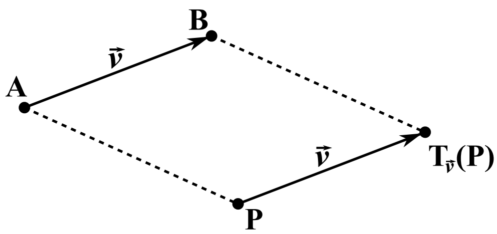


For an arbitrary $\alpha \in \mathbb{R}$ we define `the dilation` $D_{O,\alpha}$ `with centre` O `and scale factor` α analogously.

(3) Given a point O in the plane define `the rotation` $R_{O,\theta}$ `with centre` O `and angle` $0 \leq \theta < 2\pi$ to be the map which rotates the plane about O through an angle of θ in an anti-clockwise sense. More precisely, for every point $P \neq O$ define $R_{O,\theta}(P)$ to be the unique point P' on the circle with centre O and radius $|OP|$ such that $\angle POP' = \theta$.



(4) Given a vector $\vec{v} = \vec{AB}$ (an oriented line segment) define `the translation` $T_{\vec{v}}$ `by` $\vec{v}$ to be the map which translates every point in the plane by $\vec{AB}$:



E.g. if we work in coordinate plane $\mathbb{R}^2$ then given a vector $\vec{v} = (v_1, v_2)$ define

$$T_{\vec{v}} \colon \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x, y) \mapsto (x + v_1, y + v_2)$$

Of course, an abstract map from the plane to itself is not guaranteed to preserve geometrical figures. It can map a triangle to something which doesn't in the least resemble one. If we want to use maps to make precise our intuitive notion of "moving one figure on top of the other" we must demand for a map to preserve geometrical properties of a figure, that is - lengths and angles. It turns out, that is sufficient to demand that a map preserves distances, for then it necessarily preserves angles:

Definition 5. A map $f\colon$ Plane $\to$ Plane is said to be `an isometry` [5] or `distance-preserving` if for any two points A and B in the plane we have

$$d(A,B) = d(f(A), f(B)).$$

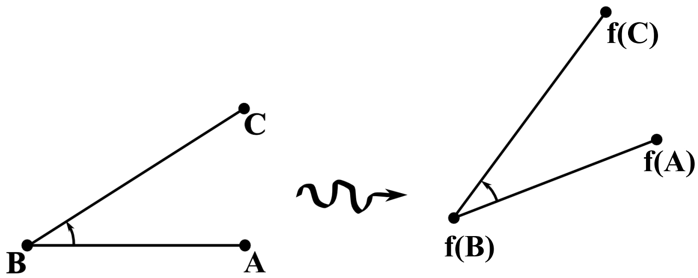
Question: Which of the four types of maps in Example 1 are isometries?

Answer: Reflections, rotations and translations are. Dilations are clearly not, with exception of dilations of scale factor 1 which are just identity maps.

Lemma 2 ("Isometries preserve angles")**.** *Let $f\colon$ Plane $\to$ Plane be an isometry. Then there are two possible cases:*

(1) *For any three points A, B and C in the plane we have*

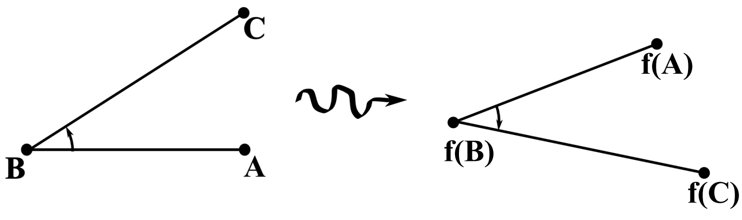
$$\angle ABC = \angle f(A)f(B)f(C).$$



We call such isometries `orientation-preserving`.

(2) For any three points A, B and C in the plane we have

$$\angle ABC = \angle f(C)f(B)f(A).$$



We call such isometries `orientation-reversing`.

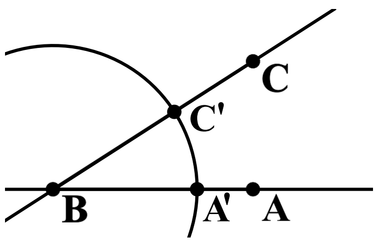
Proof. In a nutshell, the proof goes like this: isometries preserve lengths, therefore they take unit circles to unit circles. An angle was defined in terms of a length of an arc going counter-clockwise on a unit circle. An isometry, having to preserve lengths, must take this arc to an arc of the same length on the image circle. This new arc either still

[5] The word 'isometry' is derived from Greek ισομετρία (isometria) which means "of equal measure".

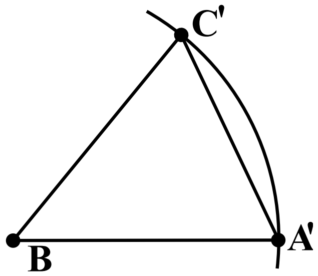
goes counter-clockwise (Case 1 above) or it now goes clockwise instead (Case 2 above). Q.E.D.

"But wait!", a skeptical reader might exclaim here. "You only know that isometries preserve distances between points. These are the lengths of *straight lines*. Why do isometries have to preserve the lengths of *circle arcs* too?" This is because we can actually approximate any circle arc by a collection of straight line segments and by increasing the number of used segments we can approximate with any necessary precision. Settling this technical point takes up most of the proof below.

And so: just as in Section 3 consider a unit circle around B, let A' be the point where it intersects line BA on the same side of B as A, and similarly for C':



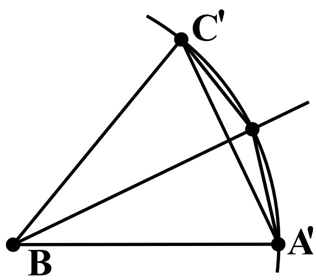
By definition, the value of $\angle ABC$ is the arc length $|\overset{\frown}{A'C'}|$. As our first and crudest approximation of this arc we take straight line segment AC



So set $d_1 = |AC|$ and observe that as the shortest path between any two points is the straight line joining them we must have

$$d_1 < |\overset{\frown}{A'C'}|.$$

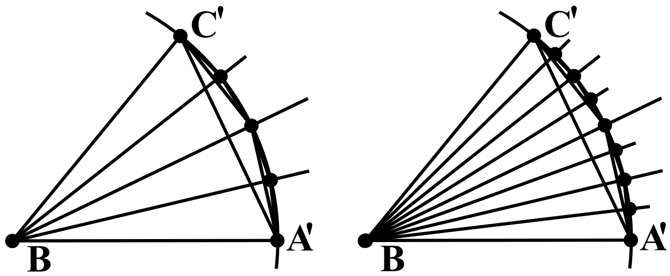
Now subdivide this arc in two equal parts and approximate each of the halves by the corresponding straight line segment:



Our second approximation of arc $|\widehat{A'C'}|$ is this two-segment path and we set d_2 to be its length, i.e. the sum of the lengths of its two segments. Observe, again, that as the shortest path between any two points is the straight line joining them we must have

$$d_1 < d_2 < |\widehat{A'C'}|.$$

We now subdivide each of the two halves of the arc in two again and obtain a four-segment approximation d_3. Then an eight-segment approximation d_4



and so on, obtaining at n-th step an 2^{n-1}-segment approximation d_n. The resulting sequence of approximations satisfies

$$d_1 < d_2 < d_3 < d_4 < \cdots < |\widehat{A'C'}|$$

and it is not too difficult to show that as n grows these multi-segment paths get arbitrarily close to the arc they approximate, in other words

$$|\widehat{A'C'}| = \lim_{n\to\infty} d_n.$$

Our isometry f takes this whole construction to an identical construction approximating the image arc from $f(A')$ to $f(C')$ on the unit circle around $f(B)$. Since f preserves the lengths of straight line segments we must have $d_n = f(d_n)$ and therefore

$$|\widehat{A'C'}| = \lim_{n\to\infty} d_n = \lim_{n\to\infty} f(d_n) = |f(\widehat{A')f(}C')|.$$

There are now two possibilities. First one is that the image arc goes counter-clockwise from $f(A')$ to $f(C')$ on the unit circle around B, and therefore its length is the value of $\angle f(A)f(B)f(C)$. We have then

$$\angle ABC = |\widehat{A'C'}| = |f(\widehat{A')f(}C')| = \angle f(A)f(B)f(C)$$

which corresponds to the Case 2 in the statement of this lemma. The other possibility is that the image arc goes clockwise from $f(A')$ from $f(C')$, i.e. it goes counter-clockwise from $f(C')$ to $f(A')$ and therefore its length is the value of $\angle f(C)f(B)f(A)$. We have then

$$\angle ABC = |\widehat{A'C'}| = |f(\widehat{A')f(}C')| = \angle f(C)f(B)f(A)$$

which corresponds to the Case 2.

It remains only to show that this behaviour must stay the same for all angles in the plane. In other words, if f is orientation-preserving on one angle, it can't be orientation-reversing on another angle and vice versa. This is a simple continuity argument: the behaviour of f can't change if we shift each of the points A, B and C by some distance which is small compared to the size of the triangle ABC. This is because if the behaviour of f does change, then the point $f(C)$ would suddenly "jump" to the other side of line $f(A)f(B)$ - and being distance-preserving f can only shift $f(A)$, $f(B)$ and $f(C)$ by the same small distances we've shifted A, B and C by. But we can now keep doing this again and again, until we've moved A, B and C to any three points in the plane we like. Therefore the behaviour of f must be the same for any triple of points in the plane. Q.E.D. □

We now see that isometries are the transformations of our plane which preserve the properties of geometrical objects. We can therefore make precise our intuitive notion that two objects are equal if we can "superimpose one upon the other and match them up exactly":

Definition 6. We say that two geometrical figures are `congruent` if there exists an isometry taking one to the other. We denote this relation by symbol $\cong$. If we specify a vertex order on the figures then we demand for an isometry to preserve it, e.g. $\triangle ABC \cong \triangle A'B'C'$ only if there exists an isometry taking A to A', B to B' and C to C'.

But now we run into a problem: clearly any two points in the plane, or any two line segments of the same length, are "equal" and should therefore be congruent to each other. But how do we show that, how do we show that there exists an isometry taking one to the other? For that matter, what is there to say that any non-trivial isometries (i.e. isometries which are not just identity maps) exist per se? The reflection maps or rotation maps we've defined above - we haven't yet got enough facts established to rigorously prove that they indeed preserve distances. Moreover, the usual such proofs (as we will see later on in this course) use triangle congruence criteria, and those are established with arguments along the lines of "these two line segments are of equal length, therefore superimpose one against the other..." - in other words, they implicitly assume existence of appropriate isometries.

This is one of the problems with Euclid's five axioms we've mentioned before. We need, in fact, two extra axioms telling us some basic facts about an existence of isometries:

Axiom 6. *Any two line segments of equal length are congruent: if A, B, C and D are four points in the plane such that $|AB| = |CD|$, then there exists an isometry taking A to B and C to D.*

Observe that we can do this by a combination of a translation and a rotation. First we do a translation by $\vec{AC}$, which by definition takes point A to point C. Now the points C and A coincide, while B and D lie on the same circle of radius $|AB| = |CD|$ around them. We can therefore do a rotation and match them up. Unfortunately, as discussed above, we haven't yet established that translations and rotations are isometries.

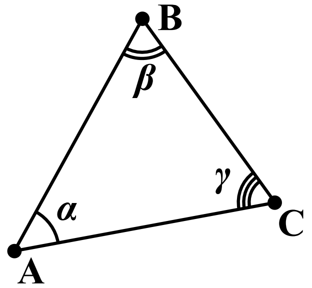
Axiom 7. *Give a line l and a point P there exists an isometry which leaves l fixed and moves P to the other side of l.*

Observe, again, that a reflection in l does precisely that. But we haven't yet established that reflections are isometries either.

5. Triangle congruences

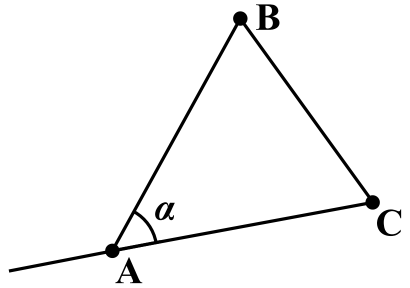
For the clarity of the exposition we assume that all our triangles are non-degenerate, that is - that their vertices are not collinear. An enthusiastic reader is encouraged to try and extend the results of this section to degenerate triangles, it is not very difficult.

Notation: In a triangle $\triangle ABC$ we denote by $\angle A$ the internal angle of the triangle at the vertex A and by α the value of $\angle A$. Similarly, we denote by β the value of $\angle B$, the internal angle at B, and by γ the value of $\angle C$, the internal angle at C:

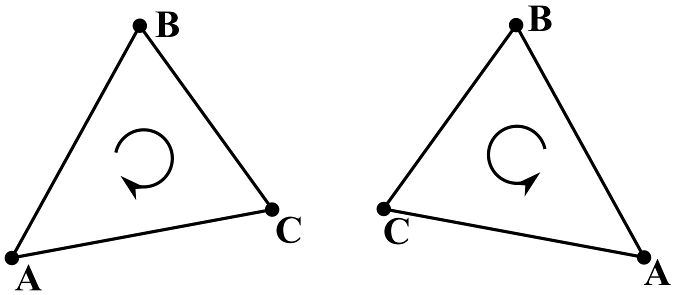


Similarly, in a triangle $\triangle A'B'C'$ we use $\angle A'$, $\angle B'$ and $\angle C'$ to denote its internal angles at A', B' and C', respectively. We further use α', β' and γ' to denote the values of these angles.

Note, that in any $\triangle ABC$ we have $0 < \alpha, \beta, \gamma < \pi$ as internal angles of a triangle are clearly less that a straight angle:



Definition 7. A triangle $\triangle ABC$ is `clockwise` (resp. `anti-clockwise`) `oriented` if moving from A to B to C takes you clockwise (resp. anti-clockwise) around the points in the interior of the triangle:



Exercise 1. Verify that an isometry $f\colon \text{Plane} \to \text{Plane}$ is orientation-preserving (resp. orientation-reversing) if and only if for every non-degenerate $\triangle ABC$ the orientation of $\triangle f(A)f(B)f(C)$ is the same as (resp. opposite to) orientation of $\triangle ABC$.

Note that if $\triangle ABC$ is clockwise oriented, then

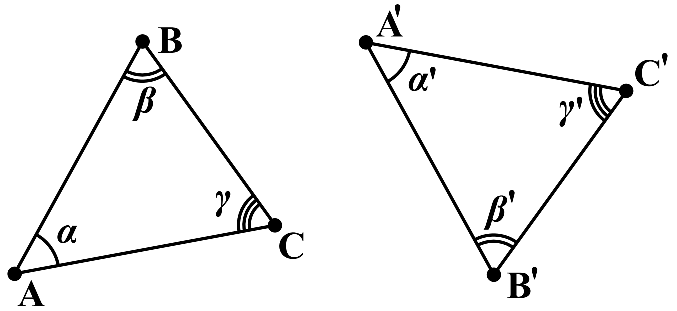
$$\alpha = \angle CAB,\ \beta = \angle ABC,\ \gamma = \angle BCA$$

and if $\triangle ABC$ is anti-clockwise oriented, then

$$\alpha = \angle BAC,\ \beta = \angle CBA,\ \gamma = \angle ACB.$$

Lemma (Lemma $2\frac{1}{2}$)**.** *Isometries preserve internal angles of a triangle. More precisely, let $\triangle ABC$ and $\triangle A'B'C'$ be a pair of triangles in a plane. If $\triangle ABC \cong \triangle A'B'C'$ then*

$$\alpha = \alpha',\ \beta = \beta',\ \gamma = \gamma'$$



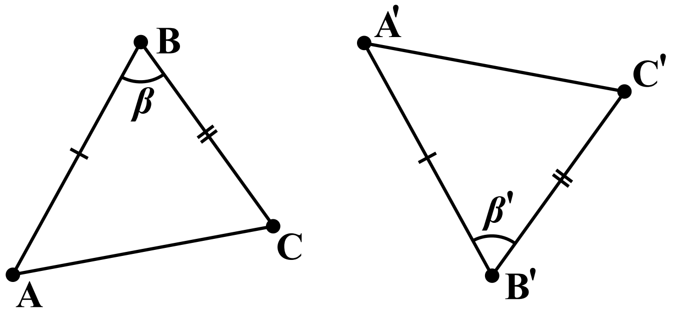
Proof. Let f be the isometry taking $\triangle ABC$ to $\triangle A'B'C'$. Observe, that the angle $f(\angle A)$ to which f takes $\angle A$ is either the internal angle at A' in $\triangle A'B'C'$ and then $f(\angle A) = \alpha' < \pi$ or the external angle at A' in $\triangle A'B'C'$ and then $f(\angle A) = 2\pi - \alpha' > \pi$.

Now apply Lemma 2. If f is orientation-preserving, then the value of $f(\angle A)$ is α and therefore $f(\angle A)$ has to be the internal angle at A' as $\alpha < \pi$. Therefore $\alpha' = f(\angle A) = \alpha$. If f is orientation-reversing (this is the case depicted on the diagram) then $f(\angle A) = 2\pi - \alpha$, and we see that $f(\angle A)$ has to be the external angle at A' as $2\pi - \alpha > \pi$. Therefore $f(\angle A) = 2\pi - \alpha'$ and we conclude that $\alpha = \alpha'$. $\square$

We now proceed to prove standard criteria for a pair of triangles to be congruent:

Lemma 3 (SAS: Side-Angle-Side)**.** *Let $\triangle ABC$ and $\triangle A'B'C'$ be a pair of triangles in the plane. If*

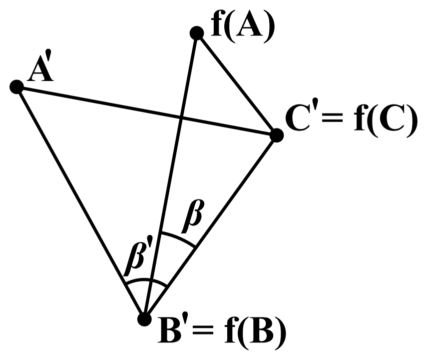
$$|BA| = |B'A'|,\ |BC| = |B'C'| \ \textit{ and } \beta = \beta'$$



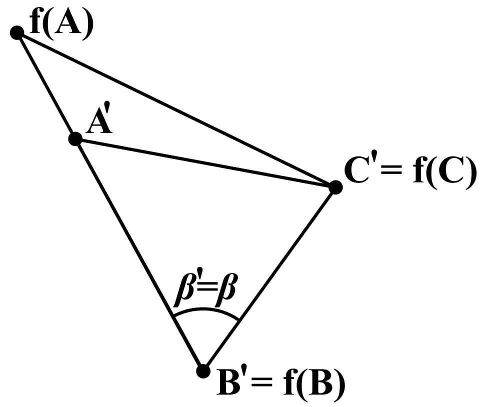
then

$$\triangle ABC \cong \triangle A'B'C'.$$

Proof. Since $|BC| = |B'C'|$ there exists by Axiom 6 an isometry f such that $f(B) = B'$ and $f(C) = C'$. By Axiom 7 we can also assume that $f(A)$ is on the same side of $B'C'$ as A'. By Lemma $2\frac{1}{2}$ the value of $\angle f(B')$ in $f(\triangle ABC)$ equals β, the value of $\angle B$ in $\triangle ABC$:

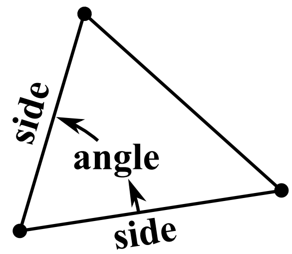


By assumption $\beta = \beta'$, therefore the values of $\angle B'$ in $\triangle f(A)B'C'$ and of $\angle B'$ in $\triangle A'B'C'$ are equal. Since A' and $f'(A)$ lie on the same side of $B'C'$ this implies that lines $B'f(A)$ and $B'A'$ actually coincide:

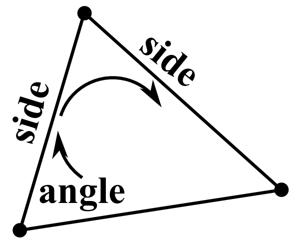


So $f(A)$ and A' on the same line through B' and on the same side of $B'C'$. By assumption we also have $|BA| = |B'A'|$ and since isometries preserve lengths, we further have $|B'f(A)| = |B'A'|$. So $f(A)$ and A' lie also at the same distance from B' and must therefore coincide. So $f(A) = A'$, $f(B) = B'$ and $f(C) = C'$. As f is an isometry, we conclude that triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent. Q.E.D. □

A warning:



SAS is a criterion for congruence of triangles.



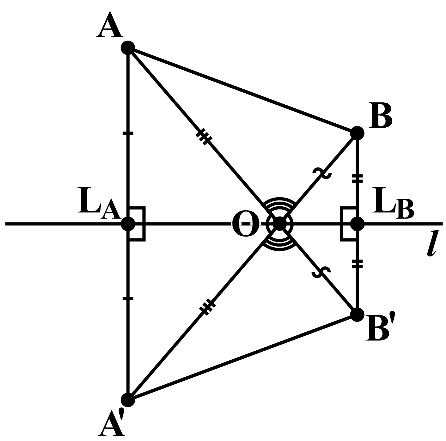
ASS is not.

Do not confuse the two.

Lemma 4. *Let l be a line in the plane. The reflection r_l in l is an isometry.*

Proof. Let A and B be a pair of points in the plane. Set $A' = r_l(A)$ and $B' = r_l(A)$. Set L_A and L_B to be the midpoints of AA' and BB'. By definition of r_l the line l is perpendicular to AA' and BB' and passes through L_A and L_B. Finally, set O to be the intersection of AB' and l.

Case 1: A and B are on the same side of l



We have $|L_BB| = |L_BB'|$ and $\angle L_B$ both in $\triangle BOL_B$ and in $\triangle B'OL_B$ equals $\pi/2$. Therefore $\triangle BOL_B \cong \triangle B'OL_B$ by SAS. Consequently

$$|OB| = |OB'| \text{ and } \angle L_BOB = \angle B'OL_B$$

Similarly, $\triangle AOL_A \cong \triangle A'OL_A$ by SAS and we have

$$|OA| = |OA'| \text{ and } \angle AOL_A = \angle L_AOA'$$

By construction AB' and l are straight lines intersecting at O. Hence $\angle AOL_A = \angle B'OL_B$ by Lemma 1, and so

$$\angle L_BOB = \angle B'OL_B = \angle AOL_A = \angle L_AOA'.$$

Let us denote the common value of these four angles by α. Since $\angle L_AOL_B$ is a straight angle we have

$$\angle BOA + 2\alpha = \angle AOL_A + \angle BOA + \angle L_BOB = \pi$$

$$\angle A'OB + 2\alpha = \angle L_AOA' + \angle A'OB' + \angle B'OL_B = \pi$$

and we conclude that $\angle BOA = \angle A'OB = \pi - 2\alpha$.

We have now established that $|OA| = |OA'|$, $|OB| = |OB'|$ and $\angle BOA = \angle A'OB$. Therefore $\triangle AOB \cong A'OB'$ by SAS and hence $|AB| = |A'B'|$. Q.E.D.

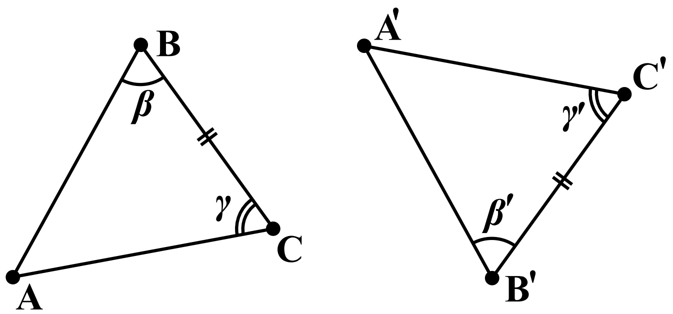
Case 2: A and B are on the opposite sides of l

Exercise! (Hint: On the diagram above

$$\angle A'OB = \angle A'OB' + \angle B'OB = \pi$$

and therefore $A'OB$ is actually a straight line). □

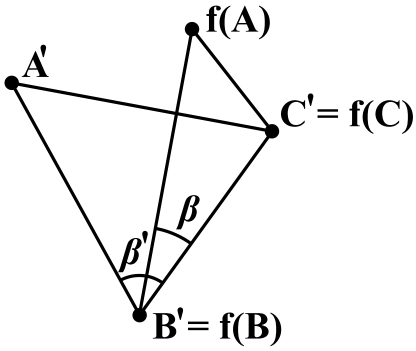
Lemma 5 (ASA: Angle-Side-Angle). *Let $\triangle ABC$ and $\triangle A'B'C'$ be a pair of triangle in the plane. If $|BC| = |B'C'|$, $\beta = \beta'$ and $\gamma = \gamma'$*



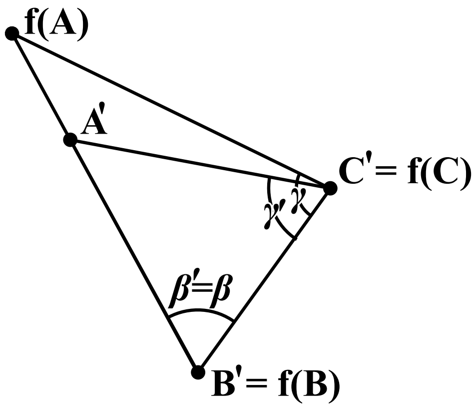
then

$$\triangle ABC \cong \triangle A'B'C'.$$

Proof. Since $|BC| = |B'C'|$ by Axioms 6 and 7 there exists an isometry f such that $f(B) = B'$, $f(C) = C'$ and $f(A)$ and A' lie on the same side of $B'C'$.

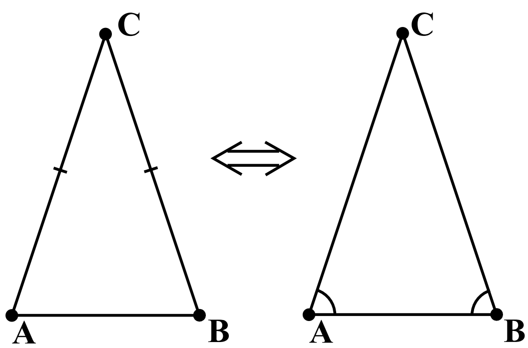


The next step is just like in Lemma 3: by assumption $\beta = \beta'$, so $B'f(A)$ and $B'A'$ make the same angle with with $B'C'$. Since $f(A)$ and A' lie on the same side of $B'C'$, it follows that they actually lie on the same line $B'A'$ through B':



Similarly, by assumption $\gamma = \gamma'$ and therefore $f(A)$ and A' also lie on the same straight line $C'A'$ through C. But now $f(A)$ and A' lie on both $B'A$ and CA'. Since two distinct straight lines can intersect at most one point A' and $f(A)$ must coincide. We have therefore $f(A) = A'$, $f(B) = B'$ and $f(C) = C'$. Since f was taken to be an isometry, we conclude that triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent. Q.E.D. □

Lemma 6. *Let $\triangle ABC$ be a triangle in the plane. Then $|AC| = |BC|$ if and only if $\alpha = \beta$.*



Proof. *"If" direction:* Suppose $\alpha = \beta$. Then $\triangle ABC \cong \triangle BAC$ by ASA, and therefore $|AC| = |BC|$.

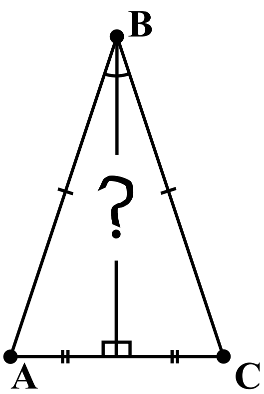
"Only if" direction: Suppose $|AC| = |BC|$. Then $\triangle ABC \cong \triangle BAC$ by SAS, and therefore $\alpha = \beta$.

Q.E.D. □

Definition 8. If a triangle has two sides which are both of the same length, it is called `an isosceles triangle`. If all three sides of a triangle are equal, it is called `an equilateral triangle`[6].

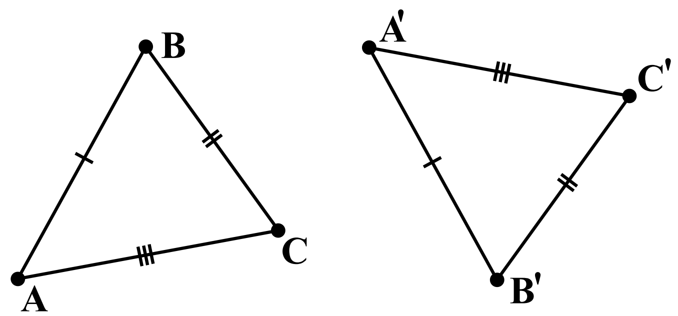
Exercise 2. Show that in an equilateral triangle all angles are equal to each other.

Lemma 7. *Let $\triangle ABC$ be an isosceles triangle with $|AB| = |CB|$. The perpendicular bisector of AC then coincides with the bisector of $\angle B$.*



Proof. Exercise! (Hint: Extend the bisector of $\angle B$ until it intersects AC. Then use triangle congruence to show that it makes right angles with AC and cuts it in half.) □

Lemma 8 (SSS: Side-Side-Side)**.** *Let $\triangle ABC$ and $\triangle A'B'C'$ be a pair of triangle in the plane. If $|AB| = |A'B'|, |BC| = |B'C|$ and $|CA| = |C'A'|$*

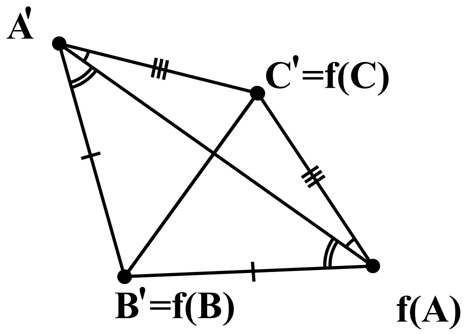


then

$$\triangle ABC \cong \triangle A'B'C'.$$

[6]The word *'isosceles'* is derived from Greek ἰσοσκελὲσ (isoskeles) which means "of equal legs". The word *'equilateral'* is derived from Latin *'aequilateralis'* which means 'of equal sides'.

Proof. By assumption $|BC| = |B'C'|$ and so by Axioms 6 and 7 there exists an isometry f such that $f(B) = B'$, $f(C) = C'$ and $f(A)$ and A' lie on the opposite sides of $B'C'$.



By assumption $|C'A'| = |CA|$ and since f is an isometry $|CA| = |f(C)f(A)|$. Therefore $\triangle A'f(A)C'$ is isosceles and by Lemma 6

$$\angle A' = \angle f(A') \quad \text{in} \quad \triangle A'f(A)C'.$$

Similarly, the assumption $|A'B'| = |AB|$ implies that $\triangle A'f(A)B'$ is isosceles and

$$\angle A' = \angle f(A') \quad \text{in} \quad \triangle A'f(A)B'.$$

Adding the above two equalities together we get

$$(\angle A' \text{ in } \triangle A'B'C') = (\angle f(A') \text{ in } \triangle f(A)B'C')\,.$$

Since also $|A'B'| = |f(A)B'|$ and $|A'C'| = |f(A)C'|$ we have by SAS

$$\triangle A'B'C' \cong \triangle f(A)B'C'.$$

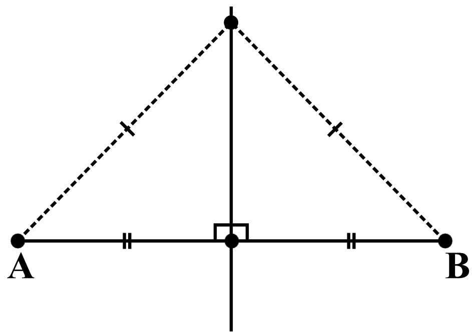
But $\triangle f(A)B'C'$ is the image of $\triangle ABC$ under isometry f, i.e.

$$\triangle f(A)B'C' \cong \triangle ABC.$$

We conclude that $\triangle A'B'C'$ and $\triangle ABC$ are congruent. Q.E.D. □

6. Circles and triangles

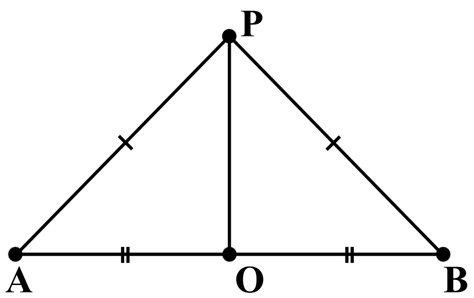
Lemma 9. *For any two points A and B in the plane the perpendicular bisector of AB is the locus*[7] *of points equidistant from A and B.*



[7]The word *'locus'* is a Latin word meaning "place" (plural: *loci*). In geometry it means a collection of all points sharing a specified property.

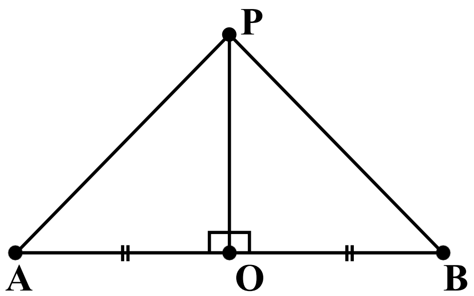
Proof. Let O be the midpoint of AB. Let P be a point in the plane.

(1) Suppose P is equidistant from A and B.



Then $\triangle APO = \triangle BPO$ by SSS. Therefore $\angle POA = \angle BOP$ and since $\angle POA$ and $\angle BOP$ add up to a straight angle, it follows that they are both right angles. So OP is perpendicular to AB and by construction O is the midpoint of AB. Therefore OP is the perpendicular bisect of AB. Q.E.D.

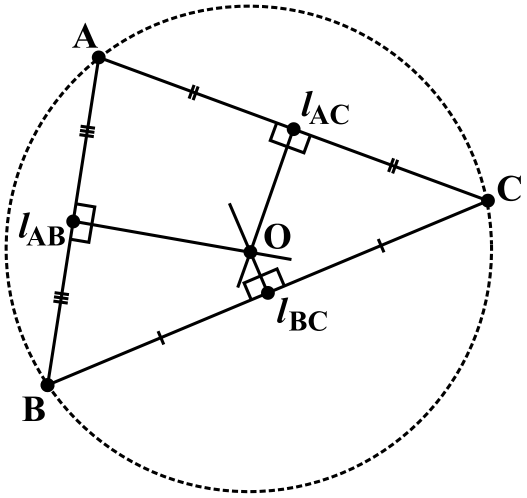
(2) Suppose P lies on the perpendicular bisect of AB.



Then $\triangle APO = \triangle BPO$ by SAS. Therefore $|AP| = |BP|$ i.e. P is equidistant from A and B. Q.E.D.

□

Theorem 10. *In any $\triangle ABC$ the perpendicular bisectors of its three sides are concurrent, i.e. meet at a point. The point O where they meet is the centre of the unique circle which passes through A, B and C.*



Proof. Let l_{AB}, l_{BC} and l_{AC} be the perpendicular bisectors of AB, BC and AC, respectively. Then by Lemma 9

$$l_{AB} = \{ \text{ the locus of points equidistant from } A \text{ and } B \ \}$$
$$l_{BC} = \{ \text{ the locus of points equidistant from } B \text{ and } C \ \}$$
$$l_{AC} = \{ \text{ the locus of points equidistant from } A \text{ and } C \ \}.$$

Let O be the unique point where l_{AB} and l_{BC} intersect. Then, by above, O is equidistant from A and B, and also O is equidistant from B and C. Therefore

$$|OA| = |OB| = |OC|$$

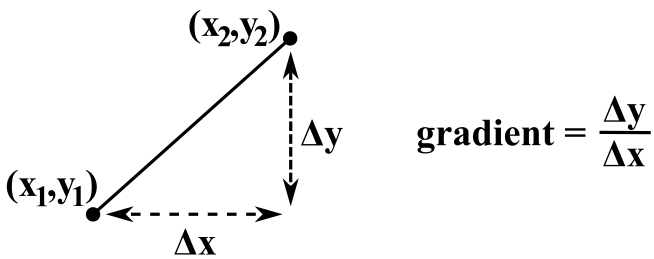
and as, in particular, $|OA| = |OC|$ point O must also lie on l_{AC}. Moreover, the circle of radius $|OA|$ and center O clearly passes through A, B and C.

Suppose there exists another circle passing through A, B and C and let O' be its center. But then O' is equidistant from A, B and C, so it belongs to each of the lines l_{AB}, l_{BC} and l_{AC}. Since any two lines intersect at no more than one point, point O' must coincide with O. Q.E.D. □

With this geometric picture in mind, we can now easily solve the problem of finding the circle passing through three given points in coordinates:

Exercise 3. Let $A = (x_1, y_1)$, $B = (x_2, y_2)$ and $C = (x_3, y_3)$ be any three points in $\mathbb{R}^2$.

(1) Find the gradient of AB and of BC.



(2) Find the midpoints L_{AB} and L_{BC} of AB and AC.

(3) Recall that the equation of a line of gradient k passing through a point (a, b) is

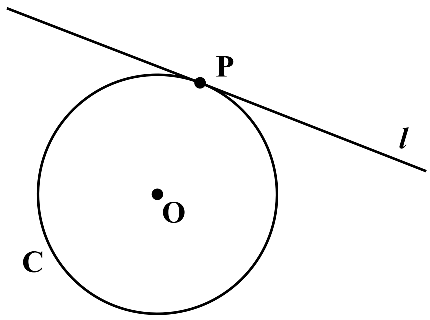
$$y = kx + (b - ka)$$

and recall that perpendicular lines have gradients whose product is -1. With this in mind, find the equations of perpendicular bisects l_{AB} and l_{BC} of AB and AC.

(4) Find the point O where l_{AB} and l_{BC} intersect. Find the distance $|OA|$. These are by Theorem 10 the center and the radius of the requisite circle.

(5) We should check that $|OA| = |OB| = |OC|$. For this, check that the formula for $|OA|$ which you've obtained in the previous step should be invariant under any permutation of indices. That is - swapping x_i with x_j and y_i with y_j shouldn't change the formula. Why is it enough to check that?

Definition 9. A line l is `tangent to a circle` C `at point` P if it meets C only at P.

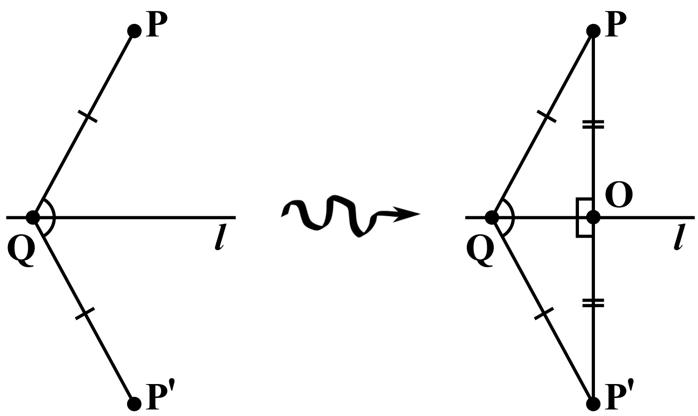


Lemma 11. *Let l be a line and P a point not on l. Then there exists a unique line through P which is perpendicular to l, and the shortest distance from P to l is along this line.*

Proof. **Existence:** Let Q be any point on l. Let P' be the unique point such that

$$\angle l, QP = \angle QP', l \quad \text{and} \quad |QP| = |QP'|.$$

Then draw the straight line segment PP' and let O be the point where PP' intersects l. We then have $\triangle OQP \cong \triangle OQP'$ by SAS



and therefore

$$|OP| = |OP'| \quad \text{and} \quad \angle POQ = \angle QOP'.$$

By construction the angle $\angle POP'$ is straight, so we have also

$$\angle POQ + \angle QOP' = \pi$$

and it follows that

$$\angle POQ = \angle QOP' = \frac{\pi}{2}$$

i.e. line PP' is perpendicular to l.

Recall now that by construction

$$\angle OQP = \angle P'QO.$$

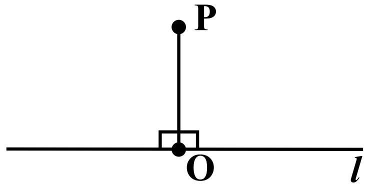
So if PQ is also perpendicular to l, then $\angle PQP' = \pi$, i.e. PQP' is in fact a straight line. But then O and Q must coincide, as otherwise lines PP' and l would intersect in two distinct points. Since Q was taken to be any point on l, it shows O is the unique point on l such that PO is perpendicular to l. This demonstrates that the line PP' is the unique line through P perpendicular to l.

Finally, observe that if Q and O are distinct, then we must have

$$2|PQ| = |PQ| + |QP'| > |PO| + |OP'| = 2|PO|$$

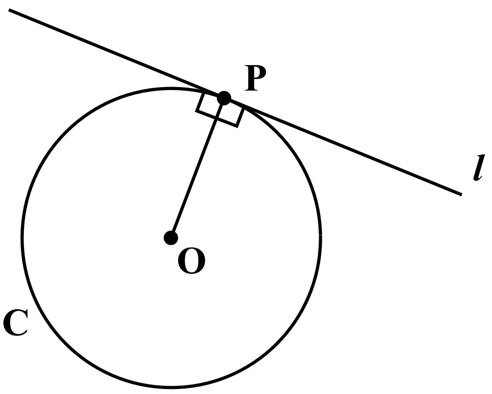
as the shortest distance between two points is the straight line. Since Q was taken to be any point on l, this shows that the distance from P to O is indeed smaller that from P to any other point on l. Q.E.D. □

Definition 10. The `distance from a point` P `to a line` l is 0, if $P \in l$, and is $|PO|$ for the unique $O \in l$ with $PO \perp l$, otherwise.

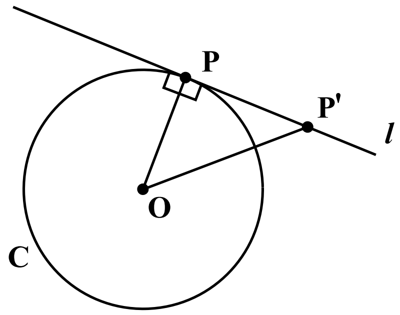


I.e. it is the shortest distance from P to l along a straight line.

Lemma 12. *Through any point P on a circle C there exists a unique line tangent to C. It is the line perpendicular to OP, where O is the center of the circle.*

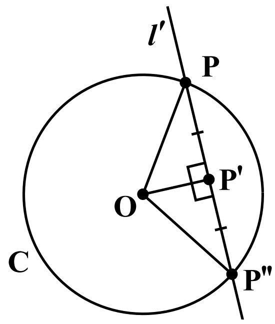


Proof. Let l be the unique line which passes through P and is perpendicular to OP. Let P' be any point of l distinct from P.



By Lemma 11 we have $|OP'| > |OP|$ and therefore P can't lie on C since $|OP|$ is the radius of C. Therefore l intersects C only at P, i.e. l is tangent to C at P.

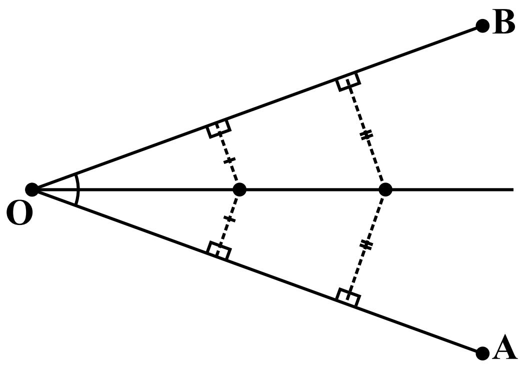
On the other hand, let l' be any line through P which is not perpendicular to OP. Then, by Lemma 11 there exists $P' \in l$, distinct from P, such that $OP' \perp l$. Let P'' be the reflection of P in line OP'.



Then line OP' is the perpendicular bisect of PP''. By Lemma 9 it is then the locus of all the points equidistant from P and P''. In particular, O is equidistant from P and P'', and so P'' lies on C. But by construction $\angle PP'P''$ is a straight angle, i.e. $P'' \in l$. So l' meets C in two points, P and P'', and is not therefore tangent to C.

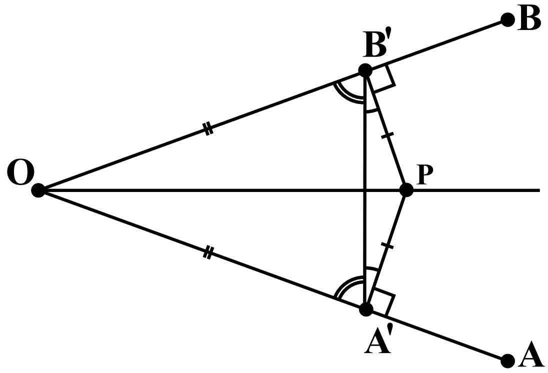
Thus the unique line through P tangent to C is the line perpendicular to OP. Q.E.D. □

Lemma 13. *Let O, A and B be any three non-collinear points in the plane. The bisector $\angle AOB$ is the locus of all points within $\angle AOB$ equidistant from lines OA and OB.*



Proof. Let P be any point within $\angle AOB$. Drop perpendiculars from P onto OA and OB: let $A' \in OA$ and $B' \in OB$ be the points such that $PA' \perp OA$ and $PB' \perp OB$. Then distances from P to OA and OB are $|PA'|$ and $|PB'|$, respectively.

(1) Suppose P is equidistant from OA and OB, i.e. $|PA'| = |PB'|$.



Then $\triangle PB'A'$ is isosceles, and therefore by Lemma 6 we have

$$\angle A'B'P = \angle PA'B'.$$

But then

$$\angle OB'A' = \frac{\pi}{2} - \angle A'B'P = \frac{\pi}{2} - \angle PA'B' = \angle B'A'O$$

and therefore $\triangle OB'A'$ is isosceles with $|OA'| = |O'B'|$. We conclude that $\triangle OB'P \cong \triangle OA'P$ by SSS and therefore

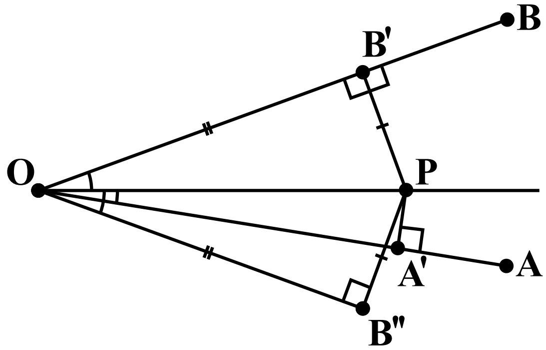
$$\angle A'OP = \angle POB'$$

i.e. OP is the bisector of $\angle AOB$.

(2) Suppose OP is the bisector of $\angle AOB$, i.e.

$$\angle A'OP = \angle POB'.$$

Then forget about point A' for a moment, and let B'' be reflection of B' in line OP.



Since the reflection in OP leaves O and P fixed, it sends $\triangle OPB'$ to $\triangle OPB''$. So these two triangles are congruent and we have

$$\angle POB' = \angle B''OP$$
$$\angle PB''O = \angle OB'P = \frac{\pi}{2}$$
$$|PB''| = |PB'|.$$

We then have

$$\angle A'OP = \angle POB' = \angle B''OP$$

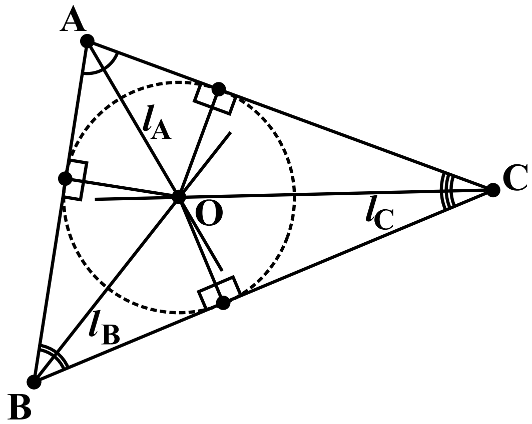
and therefore straight lines OA' and OB'' actually coincide. But then points A' and B'' must also coincide, by uniqueness of the perpendicular from P onto the line $OA'B''$ (Lemma 11). Hence

$$|PA'| = |PB''| = |PB'|$$

i.e. P is equidistant from lines OA and OB.

We conclude that OP is the bisector of $\angle AOB$ if and only if P is equidistant from OA and OB. Q.E.D. □

Theorem 14. *Let $\triangle ABC$ be a triangle in the plane. The bisectors l_A, l_B and l_C of angles $\angle A$, $\angle B$ and $\angle C$ in $\triangle ABC$ are concurrent. The point O in which they intersect is the center of the unique circle which is contained in $\triangle ABC$ and is tangent to all three of its sides.*



Proof. Let O be the point where l_A and l_B intersect. By Lemma 13:

$l_A = \{$ the locus of all points inside $\angle A$ equidistant from AB and AC $\}$

$l_B = \{$ the locus of all points inside $\angle B$ equidistant from AB and BC $\}$

$l_C = \{$ the locus of all points inside $\angle C$ equidistant from BC and AC $\}$

So $l_A \cap l_B$ is the locus of all points inside $\angle A \cap \angle B = \triangle ABC$ equidistant from AB, BC and AC. But $l_A \cap l_B$ consists only of point O. So O is the unique point within $\triangle ABC$ equidistant from all three sides of the triangle. In particular, O is equidistant from BC and AC. It must therefore also lie on l_C. So the three bisectors l_A, l_B and l_C are concurrent at O.

Let C be the circle whose center is O and whose radius is the distance r from O to the three sides of the triangle. By Lemma 11 each side of $\triangle ABC$ has only one point whose distance from O is r – the unique point where the perpendicular from O falls onto the side in question. So C meets each of the sides of $\triangle ABC$ at precisely one point, i.e. it is tangent to each of them.

For the final claim, let there be a circle C' whose center O' lies within $\triangle ABC$ and which is tangent to all three sides of $\triangle ABC$. By Lemma 12 the radius from O' to any of the three points where C' touches a side of the triangle is perpendicular to the side in question. So, by definition of distance from a point to the line, the distance from O' to each of the sides of $\triangle ABC$ is the radius of C'. Hence O' is equidistant from all three sides of $\triangle ABC$, i.e. O' is actually the point O constructed above and C' is the circle C. □

Exercise 4 (Optional; hard!)**.** In 3-dimensional space:

(1) What does it mean to say that a line through a point on a plane is perpendicular to the plane in question? Is such line unique?
(2) How would you define a distance between a point and a plane?
(3) What does it mean to say that a sphere is tangent to a plane?
(4) Can you think of how to find a sphere tangent to 4 given planes?

7. Spherical Geometry

Nowhere[8] in the results we've proved so far we've made use of Euclid's 5th axiom or of a number of statements equivalent to it, e.g. "through every point not lying on a given line there passes exactly one line parallel to the given one" or "the interior angles of every triangle in the plane add up to π".

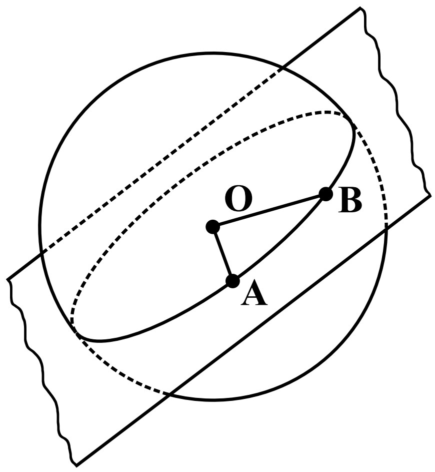
This is because it is perfectly possible to have a consistent geometrical theory where this is *not* true. It is worth, therefore, differentiating those results whose proof doesn't require the use of Euclid's 5th from those results whose proof does. Euclid himself must have been aware of this, on some level, for in "Elements" he proves as many Propositions as he can without using the 5th axiom before first invoking it in the proof of his Proposition 29.

In this section we look at one of these non-Euclidean geometries - the geometry of the surface of a sphere.

Problem: Evidently, there are no *straight* lines on the surface of the sphere. **Idea:** In Euclidean geometry the straight line joining two points A and B is the shortest path from one to the other. On the surface of a sphere the shortest path between two points is a path which goes along one of the *geodesics*.

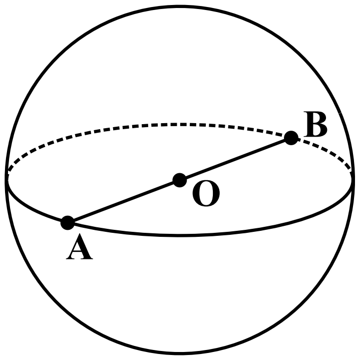
Definition 11. A `great circle` or a `geodesic` on a sphere is the intersection of this sphere with a plane passing through its centre.

[8]Almost nowhere.



Clearly a great circle is determined uniquely by the plane which carves it out on a sphere. On the other hand, in 3-dimensional Euclidean space through any three non-collinear points there passes a unique plane. So, conversely, any great circle determines uniquely the plane through the center of a sphere which carves it out.

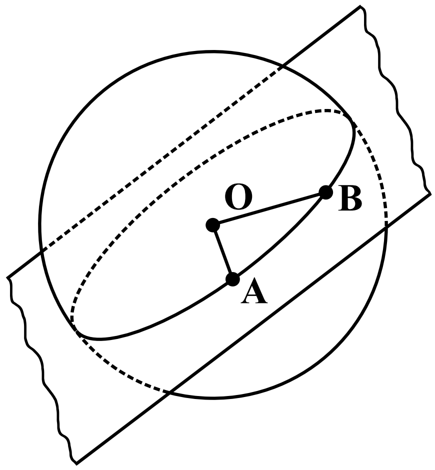
Definition 12. Two points on a sphere are called `antipodal` if they lie on the opposite ends of a diameter of the sphere.



Exercise 5. Another way of defining a great circle would be to say, that it is a circle which lies on the sphere and whose radius equals to the radius of the sphere. To check this, show that any circle lying on a sphere of radius r has radius $\leq r$, and that the equality is achieved only if the center of the circle coincides with the centre of the sphere.

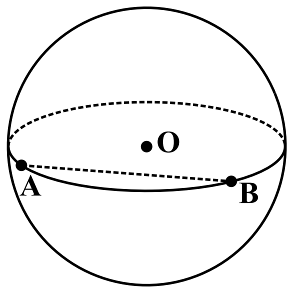
Hint: A sphere of radius r is the locus of points whose distance from its centre is r. If we have a circle lying on this sphere, take any diameter of this circle and then use the triangle equality to get an upper bound on its length. When is this upper bound achieved?

Lemma (S1)**.** *Through any two non-antipodal points on a sphere there passes a unique great circle.*



Proof. Let O denote the centre of the sphere. If two points A and B are not antipodal, then AB doesn't contain O, i.e. A, B and O are not collinear. Therefore there is a unique plane passing through O, A and B, and this plane carves out on the sphere the unique great circle which passes through A and B. □

Given any two points A and B on the sphere, we now have two notions of distance:



(1) Ambient notion: the distance from A to B is the length of a straight line joining them, i.e. the length of the shortest path from A to B in the 3-dimensional Euclidean space containing the sphere ("Tunneler's distance").
(2) Intrinsic notion: the distance from A to B is the length of the smaller (if they are non-equal) of the two arcs of any great circle joining A and B, i.e. the length of the shortest path from A to B along the surface of the sphere ("Sailor's distance").

We adopt the second one of these notions. There is not much of a difference between the two, actually: one uniquely determines the other via a simple trigonometric formula.

Definition 13. A map $f\colon \text{Sphere} \to \text{Sphere}$ is an isometry if it preserves distances between points.

Note that if an isometry of the ambient 3-space fixes the centre of the sphere, then it must take any point on the sphere to another point on the sphere, i.e. it restricts to an isometry of the sphere.

Example 2. (1) Any rotation about an axis passing through the center of the sphere yields an isometry of the sphere.
(2) Any reflection in a plane passing through the centre of the sphere yields an isometry of the sphere.
(3) Translations have no fixed points, therefore no translation of the ambient 3-space restricts to an isometry of the sphere.

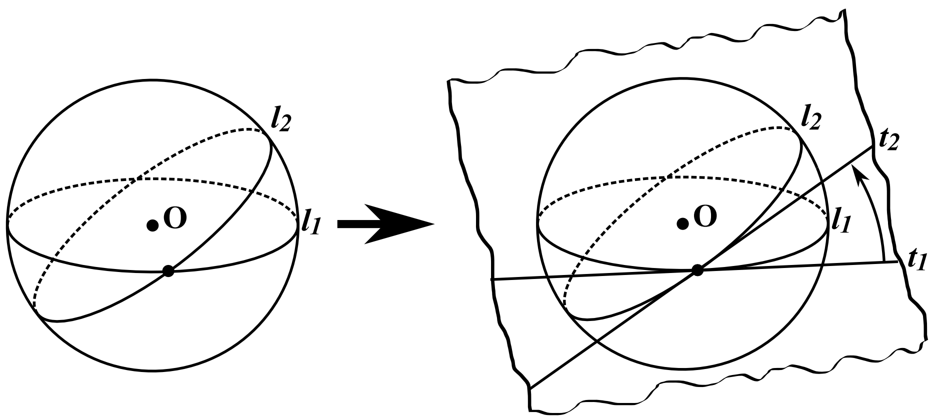
Lemma (S2). *If AB and CD are arcs of great circles which have the same length, then there exists an isometry of the sphere taking AB to CD.*

Proof. Exercise. Hint: First find an isometry which takes a great circle containing AB to a great circle containing CD. Then spin the latter around. □

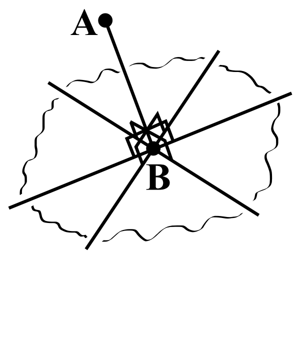
As noted before, there is no notion of parallel lines in spherical geometry - every pair of great circles meet. Another interesting point: the sum of angles in a spherical triangle is *never* π.

Question: How do we measure an angle between two spherical lines?

Idea: Measure it in the *tangent plane*:

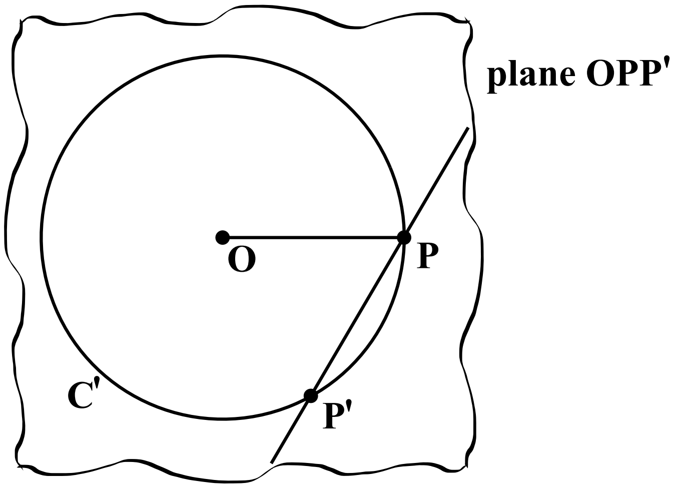


Fact: In 3-space, for any given line segment AB all the lines through B which are perpendicular to AB form together a unique plane. This plane is called the `plane through` B `perpendicular to` AB.



Lemma (S3)**.** *Let S be a sphere in a 3-dimensional space, let O be its center and P be a point on the surface of S. Then the plane through P perpendicular to OP is the unique* `plane tangent to` S `at` P(i.e. the plane which meets S only at P).

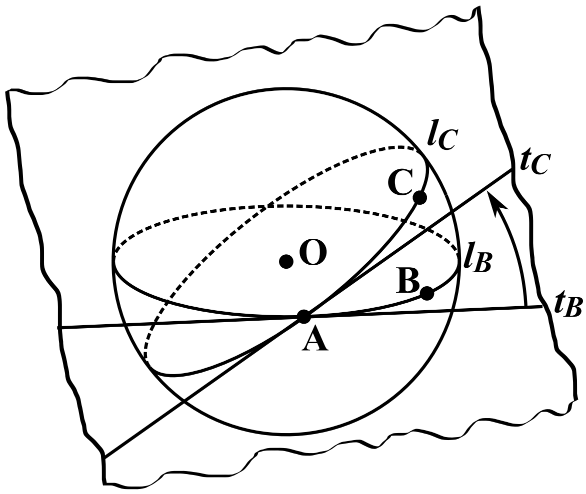
Proof. Let π_P be the plane through P perpendicular to OP. Suppose π_P meets S at some point $P' \neq P$. Then let C' be the great circle carved out on S by the plane OPP':



By construction, line PP' meets C' in two points – P and P'. On the other hand, line PP' lies in the plane π_P perpendicular to OP and is therefore itself perpendicular to OP. By Lemma 12 this makes PP' the unique line tangent to C' at P, i.e. PP' meets C' only at point P. This is a contradiction, and so π_P meets S only at P.

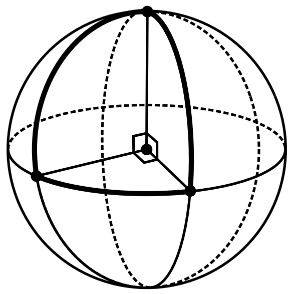
Conversely, let π' be any plane tangent to S at P. Let l be any line through P in π'. Let C_l be the great circle carved out on S by the plane containing l and O. By assumption, π' is tangent to S at P. Therefore any line in π' through P is tangent to any great circle on S through P. In particular, line l is tangent to C_l at P. By Lemma 12, line l is then perpendicular to the radius of C_l at P, i.e. $l \perp OP$. Since l was taken to be an arbitrary line through P in π', we conclude that all lines through P in π' are perpendicular to OP. So π' is the plane through P perpendicular to OP. Q.E.D. □

Definition 14. Let AB and AC be two arcs of two great circles l_B and l_C on a sphere S. Let π_A be the plane tangent to S at A and let t_B and t_C be the tangent lines carved out on π_A by the planes of l_B and l_C respectively. The spherical angle $\angle BAC$ between l_B and l_C is defined to be the planar angle $\angle t_B, t_C$ between t_B and t_C in plane π_A.



NB: In 3-dimensional Euclidean geometry there is a well-defined notion of an angle between a pair of intersecting planes. In terms of that notion, the spherical angle between two great circles on a sphere is simply the angle between the planes containing them.

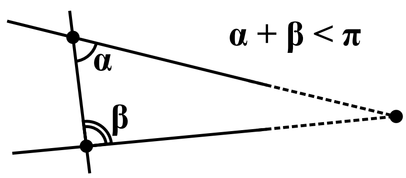
Example 3. Cut the sphere with three planes through its center which are perpendicular to each other. By above, the interior angles of the resulting spherical triangle are each equal to $\frac{\pi}{2}$. The sum of the interior angles in this triangle is therefore $\frac{3\pi}{2}$.



8. Plane geometry: the parallel postulate

We now return to plane geometry and proceed to prove the results which require the use of:

Euclid's 5th Axiom (The Parallel Postulate): *If a straight line falling on two straight lines makes the interior angles on one of the sides which are less than two right angles in total, then the two straight lines, if produced indefinitely, meet on that side.*

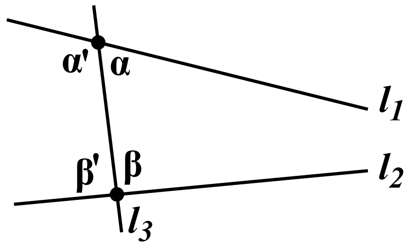


This is, in fact, equivalent to:

Playfair's Axiom: *Given a line l and a point P not on l there is a unique line through P parallel to P.*

Proof. Euclid's 5th Axiom $\Rightarrow$ Playfair's Axiom:

Let l_1 and l_2 be a pair of lines and l_3 be a third line falling upon them. Denote the internal angles l_3 makes with l_1 and l_2 as follows:



Since

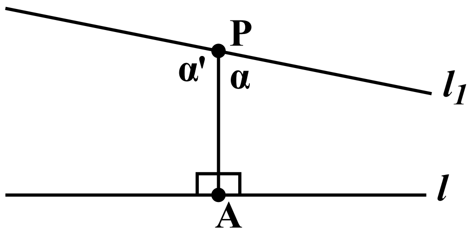
$$(\alpha' + \alpha) + (\beta' + \beta) = \pi + \pi = 2\pi$$

there are three possible cases

- $\alpha + \beta > \pi$ and $\alpha' + \beta' < \pi$
- $\alpha + \beta < \pi$ and $\alpha' + \beta' > \pi$
- $\alpha + \beta = \pi$ and $\alpha' + \beta' = \pi$

We therefore see that Euclid's 5th axiom implies that l_1 and l_2 are parallel if and only if $\alpha + \beta = \alpha' + \beta' = \pi$. In other words, two lines are parallel if and only if a line falling on them makes on either side of itself the internal angles which sum up to π.

Let now l be any line and P a point not on it. Drop a perpendicular PA from P onto l, i.e. let A be the unique point of l such that $PA \perp l$. Let now l_1 be any line through P.

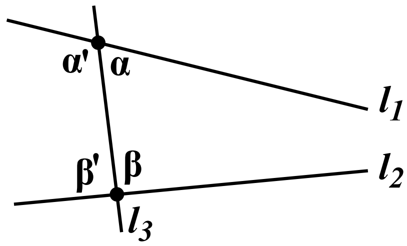


By above l_1 is parallel to P if and only if $\alpha' = \alpha = \frac{\pi}{2}$, i.e. if $l_1 \perp PA$. Since there is only one line through P perpendicular to PA, we conclude that there is only one line through P parallel to l. Q.E.D.

Playfair's Axiom $\Rightarrow$ Euclid's 5th Axiom:

Exercise (hard!). □

Lemma 15. *Let l_1 and l_2 be a pair of parallel lines. Let l_3 be a line falling on l_1 and l_2. Let the internal angles l_3 makes with l_1 and l_2 be as marked on the diagram*



then

$$\alpha' = \beta \quad and\ \alpha = \beta'.$$

Proof. Since l_1 is parallel to l_2 we must have by Euclid's 5th axiom

$$\alpha + \beta = \alpha' + \beta' = \pi.$$

But since α makes a straight angle with α', we have also

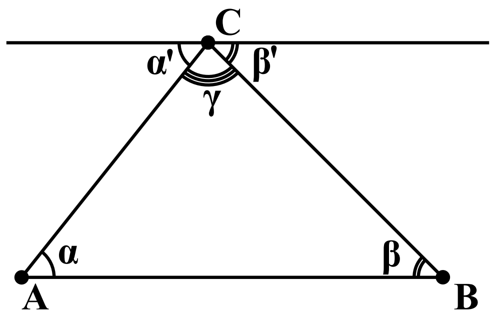
$$\alpha + \alpha' = \pi$$

and therefore $\alpha' = \beta$ and $\alpha = \beta'$. Q.E.D. □

Lemma 16. *Let $\triangle ABC$ be a triangle in the plane. Then its interior angles sum up to π, i.e.*

$$\angle A + \angle B + \angle C = \pi.$$

Proof. Let l be the unique line through C parallel to AB. Let α' be the angle l makes with AC that is adjacent to $\angle C$ in $\triangle ABC$. Similarly, let β' be the angle l makes with BC that is adjacent to $\angle C$.

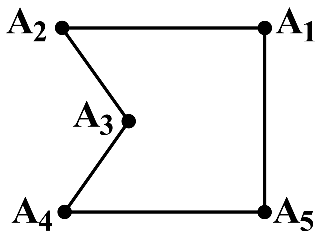


We have $\alpha' + \gamma + \beta' = \pi$ as the corresponding angles make up a straight angle. On the other hand, by Lemma 15 we have $\alpha = \alpha'$ and $\beta = \beta'$. We conclude that $\alpha + \beta + \gamma = \pi$. Q.E.D. □

There is an alternative way to prove Lemma 15, which is more general and gives a formula for the sum of interior angles in any plane n-gon:

Theorem 17. *The sum of interior angles in a plane n-gon adds up to $(n-2)\pi$.*

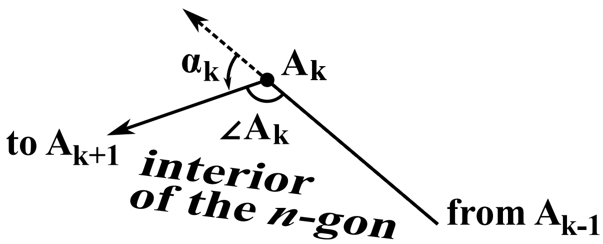
Proof. Label the vertices of the n-gon by $A_1, \ldots, A_n$ in such a way that starting at A_1 and travelling *anti-clockwise* along the circumference one encounters first A_2, then A_3, and so on.



Now imagine placing a pencil along A_1A_2 so that its tip is pointing towards A_2. Now move pencil along the circumference of the n-gon, turning it once it reaches vertex A_2 in such a way that it would then point along the next edge - A_2A_3. Repeat this until the pencil would arrive back to its starting point. Let us now sum up all the angles the pencil was turned through on its way, counting each angle with a plus sign if the pencil was turned anti-clockwise and with a minus sign otherwise. Since the pencil went once around the interior of the n-gon anti-clockwise and is now back where it started and facing the same way, the sum total is 2π.

On the other hand, at each vertex A_k we've encountered one of the two possible situations:

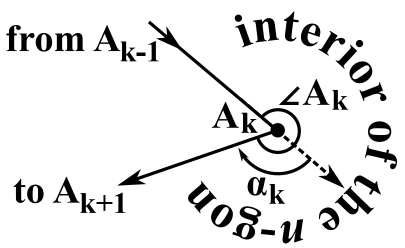
- *Case 1:* $\angle A_k < \pi$



In this case, the pencil was rotated anti-clockwise through an angle of α_k and we have:

$$\angle A_k + \alpha_k = \pi.$$

- *Case 2:* $\angle A_k > \pi$



In this case, the pencil was rotated clock-wise through an angle of α_k and we have:

$$\angle A_k - \alpha_k = \pi.$$

We see therefore that, taking account of the sign, the contribution to the total sum at each vertex A_k is $\pi - \angle A_k$. Hence

$$2\pi = \sum_{k=1}^{n} (\pi - \angle A_k) = n\pi - \sum_{k=1}^{n} \angle A_k$$

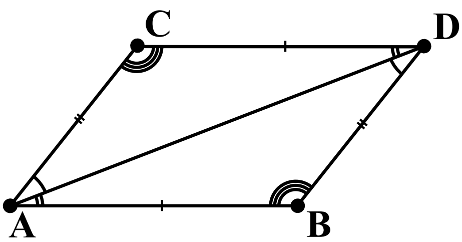
and so finally

$$\sum_{k=1}^{n} \angle A_k = (n-2)\pi.$$

Q.E.D. □

Exercise 6. Where and how does the proof of Theorem 17 make use of the parallel postulate? Hint: what does this "moving pencil" argument mean in the rigorous language we've been trying to develop?

Lemma 18. *Let $ABCD$ be a parallelogram (a quadrilateral whose sides are pairwise parallel) with $AB \parallel CD$ and $AC \parallel BD$.*



Then

$$|AB| = |CD|, \ |AC| = |BD|, \ \angle A = \angle D \ \text{ and } \angle B = \angle C.$$

Proof. By Lemma 15 we have

$$\angle BAD = \angle CDA \ \text{ and } \angle ADB = \angle DAC.$$

It follows that $\angle A = \angle D$ in $ABCD$. It also follows that $\triangle ABD \cong \triangle DCA$ by SAS. And therefore

$$|AB| = |CD|, \ |AC| = |BD| \ \text{ and } \ \angle B = \angle C \text{ in } ABCD.$$

Q.E.D. □

Exercise 7. Show that the point where the diagonals of a parallelogram intersect is the midpoint of each of the diagonals.

9. Area

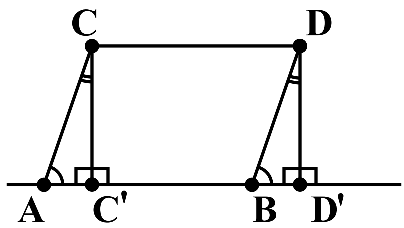
Definition 15. The area of any geometrical figure is uniquely defined by the following:

(1) The area of a rectangle is the product of the lengths of its two adjacent sides.
(2) If two figures are disjoint or meet only along their edges, then the area of their union is the sum of their areas.
(3) Congruent figures have equal areas.

NB: Essentially, items (1) and (3) could be replaced by saying that any square of side 1 has area 1, i.e. it's all about a choice of a unit. The main defining property of area is (2).

Lemma 19. *The area of a parallelogram is its base times its height, where the "base" is the length of any side of the parallelogram, and the "height" is the distance from either of the remaining two vertices to the line through the chosen side.*

Proof. Drop perpendiculars CC' and DD' from C and D onto line AB. In other words, let C' and D' be points of AB such that $CC' \perp AB$ and $DD' \perp AB$.



$$\text{"base"} = |AB|, \quad \text{"height"} = |CC'| = |DD'|$$

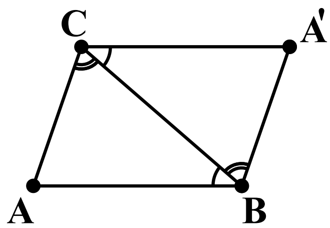
It follows from Lemma 15 that $\angle C'AC = \angle D'BD$. By construction $\angle CC'A = \angle DD'B = \frac{\pi}{2}$ and since the sum of interior angles in a triangle is π (Lemma 16) we must also have $\angle ACC' = \angle BDD'$. Finally, as the opposite sides in a parallelogram are of equal length (Lemma 18) we have $|AC| = |BD|$ and therefore $\triangle ACC' \cong \triangle BDD'$ by ASA. Hence

$$\text{area}(ABCD) = \text{area}(C'D'CD) = |C'D'| \times |CC'| = |AB| \times |CC'|.$$

Q.E.D. □

Lemma 20. *The area of a triangle is a half of its base times its height, where the "base" is the length of any side of the triangle, and the "height" is the distance from the remaining vertex to the line through the chosen side.*

Proof. Choose side AB as the "base", then the "height" is the distance from C to AB. Draw a line through B parallel to AC, a line through C parallel to AB, and let A' be the point where they intersect.



By same argument as in Lemma 18 we have $\triangle ABC \cong \triangle A'CB$. Hence

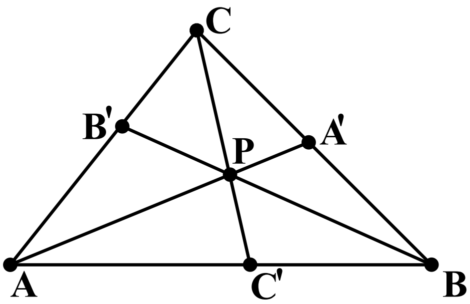
$$\text{area}(ABCA') = \text{area}(ABC) + \text{area}(A'CB) = 2 \times \text{area}(ABC).$$

On the other hand, by Lemma 19 the area of parallelogram $ABCA'$ is its base times its height. Choosing side AB as the base, we have

$$\text{area}(ABCA') = |AB| \times \text{dist}(C, AB).$$

It follows that the area of $\triangle ABC$ is $\frac{1}{2}|AB| \times \text{dist}(C, AB)$. Q.E.D. $\square$

Theorem 21 (Ceva's Theorem)**.** *Let $\triangle ABC$ be a triangle in the plane, and let A', B' and C' be points on BC, AC and AB respectively. If AA', BB' and CC' are concurrent at some point P in the interior of the triangle*



then we have

$$\frac{|AC'|}{|C'B|} \times \frac{|BA'|}{|A'C|} \times \frac{|CB'|}{|B'A|} = 1.$$

Proof. The area of a triangle is a half of its base times its height (Lemma 20) and so we have:

$$\text{area}(\triangle PAC') = \frac{1}{2}|AC'| \times \text{dist}(P, AC')$$
$$\text{area}(\triangle PBC') = \frac{1}{2}|C'B| \times \text{dist}(P, C'B)$$

Since A, C' and B are collinear, we also have

$$\text{dist}(P, AC') = \text{dist}(P, C'B)$$

and therefore

$$\frac{|AC'|}{|C'B|} = \frac{\text{area}(\triangle PAC')}{\text{area}(\triangle PBC')}.$$

Similarly we obtain

$$\frac{\text{area}(\triangle CAC')}{\text{area}(\triangle CBC')} = \frac{|AC'|}{|C'B|}.$$

It is also evident from the diagram that

$$\text{area}(\triangle CAP) = \text{area}(\triangle CAC') - \text{area}(\triangle PAC')$$
$$\text{area}(\triangle CBP) = \text{area}(\triangle CBC') - \text{area}(\triangle PBC').$$

Therefore

$$\frac{\text{area}(\triangle CAP)}{\text{area}(\triangle CBP)} = \frac{\text{area}(\triangle CAC') - \text{area}(\triangle PAC')}{\text{area}(\triangle CBC') - \text{area}(\triangle PBC')} =$$

$$= \frac{\frac{|AC'|}{|C'B|}\text{area}(\triangle CBC') - \frac{|AC'|}{|C'B|}\text{area}(\triangle PBC')}{\text{area}(\triangle CBC') - \text{area}(\triangle PBC')} = \frac{|AC'|}{|C'B|}$$

Similarly we obtain that

$$\frac{|BA'|}{|A'C|} = \frac{\text{area}(\triangle ABP)}{\text{area}(\triangle CAP)} \quad \text{and} \quad \frac{|CB'|}{|B'A|} = \frac{\text{area}(\triangle CBP)}{\text{area}(\triangle ABP)}$$

and so finally we have

$$\frac{|AC'|}{|C'B|}\times\frac{|BA'|}{|A'C|}\times\frac{|CB'|}{|B'A|} = \frac{\text{area}(\triangle CAP)}{\text{area}(\triangle CBP)}\times\frac{\text{area}(\triangle ABP)}{\text{area}(\triangle CAP)}\times\frac{\text{area}(\triangle CBP)}{\text{area}(\triangle ABP)} = 1.$$

Q.E.D. □

Exercise 8. Prove the converse to Ceva's Theorem. That is, prove that if $\triangle ABC$ is a triangle in the plane, and if A', B' and C' are points on BC, AC and AB respectively, then

$$\frac{|AC'|}{|C'B|}\times\frac{|BA'|}{|A'C|}\times\frac{|CB'|}{|B'A|} = 1$$

implies that AA', BB' and CC' are concurrent.

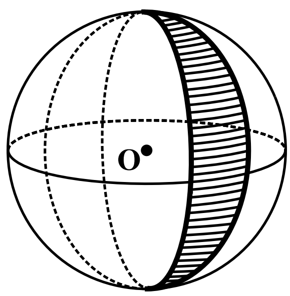
Hint: Let P be the intersection of AA' and BB'. Let C'' be the point where the line through CP intersects AB. What can we say about $\frac{|AC''|}{|C''B|}$?

Exercise 9. The medians of a triangle are the lines joining each vertex to the middle of the opposite side. Show that the three medians of a triangle are concurrent.

10. Triangles on a sphere

We assume without proof that the surface area of a sphere of radius r is $4\pi r^2$.

Definition 16. An area contained in a sector between two great circles is called a `lune`.

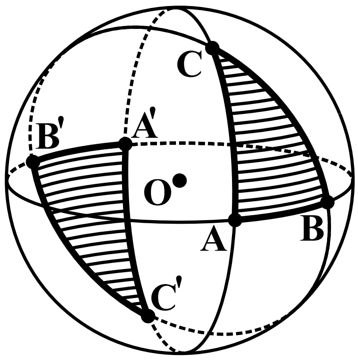


It is clear that the area of the lune is proportional to the corresponding angle between the great circles. Since the full sphere, which corresponds to an angle of 2π, has the surface area of $4\pi r^2$, we conclude that a lune of an angle θ has area $\frac{\theta}{2\pi}4\pi r^2 = 2\theta r^2$.

Theorem 22. *Let $\triangle ABC$ be a triangle on a sphere of radius r. Then*

$$\angle A + \angle B + \angle C = \pi + \frac{area(ABC)}{r^2}.$$

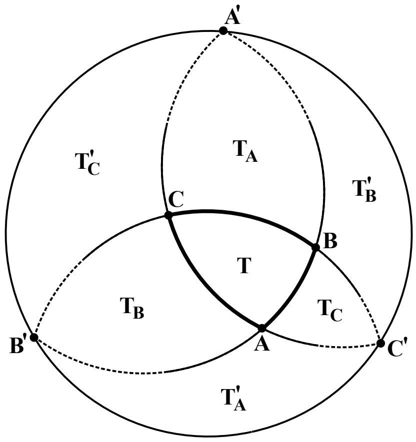
Proof. Any spherical triangle has an antipodal twin - the triangle carved out by the same great circles, but whose vertices are antipodal to the vertices of the original triangle. Clearly, the map which sends every point of a sphere to its antipode is an isometry. So any triangle is congruent to its antipodal twin. On the diagram below we've marked the antipodal twin of $\triangle ABC$ - triangle $\triangle A'B'C'$.



Altogether, the great circles AB, BC and AC divide the sphere into 8 triangles which make up 4 pairs of antipodal twins. We write:

$$T = \triangle ABC \text{ and } T' = A'B'C', \quad T_A = \triangle A'BC \text{ and } T'_A = AB'C'$$
$$T_B = \triangle AB'C \text{ and } T'_B = A'BC', \quad T_C = \triangle ABC' \text{ and } T'_C = A'B'C$$

and we write t for the area of T, t_A for the area of T_A, t'_A for the area of T'_A, et cetera. Let us now "unwrap" the picture above onto a flat plane by cutting out $\triangle A'B'C'$ and making it the triangle "at infinity":



We can see that T and T_A make up a lune of angle $\angle A$. The area of a lune of angle $\angle A$ is $2r^2\angle A$ and therefore

$$t + t_A = \angle A \cdot 2r^2$$

and similarly

$$t + t_B = \angle B \cdot 2r^2 \quad \text{and} \quad t + t_C = \angle C \cdot 2r^2.$$

We conclude that

$$3t + t_A + t_B + t_C = (\angle A + \angle B + \angle C)2r^2.$$

On the other hand

$$4\pi r^2 = \text{surface area of the sphere} = t + t_A + t_B + t_C + t' + t'_A + t'_B + t'_C.$$

As antipodal twins are congruent, we have $t = t'$, $t_A = t'_A$, etc. Hence

$$t + t_A + t_B + t_C + t' + t'_A + t'_B + t'_C = 2(t + t_A + t_B + t_C)$$

and so finally

$$t + t_A + t_B + t_C = 2\pi r^2.$$

We conclude that

$$2t = (3t + t_A + t_B + t_C) - (t + t_A + t_B + t_C) =$$
$$= (\angle A + \angle B + \angle C - \pi)2r^2$$

and therefore

$$\angle A + \angle B + \angle C = \pi + \frac{t}{r^2}.$$

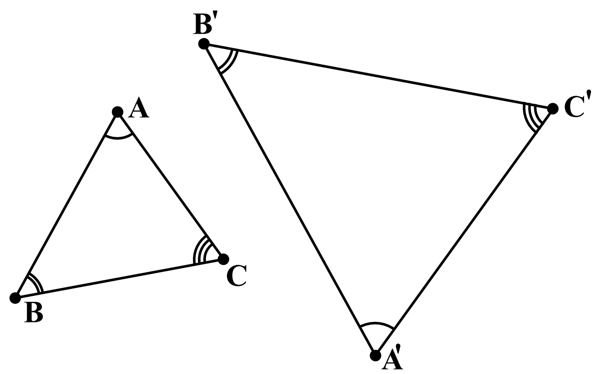
Q.E.D. □

Corollary 23. *In a non-degenerate spherical triangle (a triangle with non-zero area) the sum of interior angles is strictly greater than π.*

11. Similarity

Definition 17. Let $\triangle ABC$ and $\triangle A'B'C'$ be a pair of triangles in the plane. We say that $\triangle ABC$ is `similar` to $\triangle A'B'C'$, and write $\triangle ABC \sim \triangle A'B'C'$, if

$$\angle A = \angle A', \ \angle B = \angle B' \text{ and } \angle C = \angle C'.$$



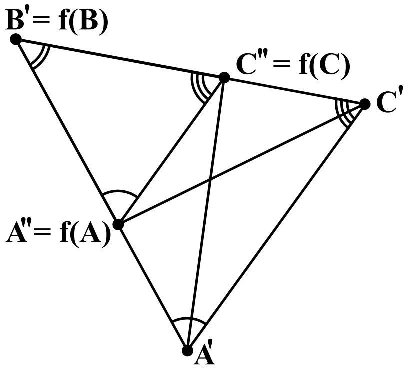
Lemma 24. *Let $\triangle ABC$ and $\triangle A'B'C'$ be a pair of triangles in the plane. If $\triangle ABC \sim \triangle A'B'C'$ then*

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|AC|}{|A'C'|}.$$

Proof. Let us prove that

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|},$$

the other two equalities are proved analogously. Since $\angle B = \angle B'$ there exists an isometry f which takes B to B', takes C to some point C'' on the line $B'C'$ on the same side of B' as C' and takes A to some point A'' on the line $B'A'$ on the same side of B' as A'.



By assumption $\angle BCA = \angle BC'A'$ and therefore line $C''A''$ is parallel to the line $C'A'$. Therefore $\mathrm{dist}(C', C''A'') = \mathrm{dist}(A', C''A''')$ and, as the area of a triangle is its base time its height, (Lemma 20)

$$\begin{aligned}\mathrm{area}(C'C''A'') &= \frac{1}{2}|C''A''| \times \mathrm{dist}(C', C''A'') = \\ &= \frac{1}{2}|C''A''| \times \mathrm{dist}(A', C''A'') = \mathrm{area}(A'C''A'').\end{aligned}$$

Using Lemma 20 again we obtain

$$\frac{\mathrm{area}(C'A''B')}{\mathrm{area}(A'B'C')} = \frac{\frac{1}{2}|A''B'| \times \mathrm{dist}(C', A'B')}{\frac{1}{2}|A'B'| \times \mathrm{dist}(C', A'B')} = \frac{|A''B'|}{|A'B'|} = \frac{|AB|}{|A'B'|}$$

$$\frac{\mathrm{area}(A'B'C'')}{\mathrm{area}(A'B'C')} = \frac{\frac{1}{2}|B'C''| \times \mathrm{dist}(A', B'C')}{\frac{1}{2}|B'C'| \times \mathrm{dist}(A', B'C')} = \frac{|B'C''|}{|B'C'|} = \frac{|BC|}{|B'C'|}.$$

We therefore conclude that

$$\begin{aligned}\frac{|AB|}{|A'B'|} &= \frac{\mathrm{area}(C'A''B')}{\mathrm{area}(A'B'C')} = \frac{\mathrm{area}(C'A''C'') + \mathrm{area}(A''C''B')}{\mathrm{area}(A'B'C')} = \\ &= \frac{\mathrm{area}(A'A''C'') + \mathrm{area}(A''C''B')}{\mathrm{area}(A'B'C')} = \frac{\mathrm{area}(A'C''B')}{\mathrm{area}(A'B'C')} = \frac{|BC|}{|B'C'|}.\end{aligned}$$

Q.E.D. □

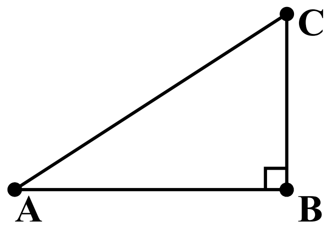
Exercise 10. Prove the converse to Lemma 24. That is, prove that if $\triangle ABC$ and $\triangle A'B'C'$ are a pair of triangles such that

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|AC|}{|A'C'|}$$

then

$$\triangle ABC = \triangle A'B'C'.$$

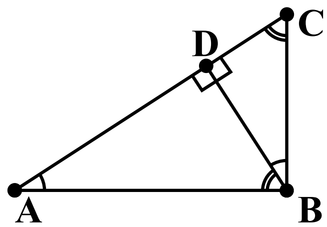
Theorem 25 (Pythagoras Theorem)**.** *Let $\triangle ABC$ be a triangle in the plane such that $\angle B$ is a right angle,*



then

$$|AB|^2 + |BC|^2 = |AC|^2.$$

Proof. Drop a perpendicular BD from B onto AC.



Then since the sum of interior angles of a triangle is π we have

$$\angle BAD + \angle DCB = \frac{\pi}{2}$$
$$\angle BAD + \angle ADB = \frac{\pi}{2}$$
$$\angle CBD + \angle DCB = \frac{\pi}{2}$$

and so

$$\angle BAD = \angle CBD \quad \text{and} \quad \angle DCB = \angle ADB.$$

By definition of similarity of triangles we have

$$\triangle ABD \cong \triangle ABC \cong \triangle BCD.$$

Therefore by Lemma 24 we have

$$\frac{|AD|}{|AB|} = \frac{|AB|}{|AC|}$$

and consequently

$$|AB|^2 = |AD||AC|.$$

Similarly,

$$|BC|^2 = |DC||AC|$$

and we conclude that

$$|AB|^2 + |BC|^2 = (|AD| + |DC|)|AC| = |AC|^2.$$

Q.E.D. □

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